1. Sort the letters: A BB II L O P R T Y. There are 11 letters in all. We build arrangements by starting with 11 ‘slots’ and placing the letters in these slots, e.g

$$\text{A B I B I L O P R T Y}$$

Create an arrangement in stages and count the number of possibilities at each stage:

Stage 1: Choose one of the 11 slots to put the A: \(\binom{11}{1}\)

Stage 2: Choose two of the remaining 10 slots to put the B’s: \(\binom{10}{2}\)

Stage 3: Choose two of the remaining 8 slots to put the B’s: \(\binom{8}{2}\)

Stage 4: Choose one of the remaining 6 slots to put the L: \(\binom{6}{1}\)

Stage 5: Choose one of the remaining 5 slots to put the O: \(\binom{5}{1}\)

Stage 6: Choose one of the remaining 4 slots to put the P: \(\binom{4}{1}\)

Stage 7: Choose one of the remaining 3 slots to put the R: \(\binom{3}{1}\)

Stage 8: Choose one of the remaining 2 slots to put the T: \(\binom{2}{1}\)

Stage 9: Use the last slot for the Y: \(\binom{1}{1}\)

Number of arrangements:

\[
\binom{11}{1}\binom{10}{2}\binom{8}{2}\binom{6}{1}\binom{5}{1}\binom{4}{1}\binom{3}{1}\binom{2}{1}\binom{1}{1} = 11 \cdot \frac{10 \cdot 9}{2} \cdot \frac{8 \cdot 7}{2} \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 9979200
\]

Note: choosing 11 out of 1 is so simple we could have immediately written 11 instead of belaboring the issue by writing \(\binom{11}{1}\). We wrote it this way to show one systematic way to think about problems like this.

2. Build the pairings in stages and count the ways to build each stage:

Stage 1: Choose the 4 men: \(\binom{6}{4}\).

Stage 2: Choose the 4 women: \(\binom{7}{4}\).

We need to be careful because we don’t want to build the same 4 couples in multiple ways. Line up the 4 men \(M_1, M_2, M_3, M_4\)

Stage 3: Choose a partner from the 4 women for \(M_1\): 4.

Stage 4: Choose a partner from the remaining 3 women for \(M_2\): 3

Stage 5: Choose a partner from the remaining 2 women for \(M_3\): 2

Stage 6: Pair the last women with \(M_4\): 1
Number of possible pairings: $\binom{6}{4} \binom{7}{4} 4!$.

Note: we could have done stages 3-6 in on go as: Stages 3-6: Arrange the 4 women opposite the 4 men: 4! ways.

3. We are given $P(A^c \cap B^c) = \frac{2}{3}$ and asked to find $P(A \cup B)$.

$A^c \cap B^c = (A \cup B)^c \Rightarrow P(A \cup B) = 1 - P(A^c \cap B^c) = \frac{1}{3}.

4. $D$ is the disjoint union of $D \cap C$ and $D \cap C^c$.

So, $P(D \cap C) + P(D \cap C^c) = P(D)$

$\Rightarrow P(D \cap C^c) = P(D) - P(D \cap C) = 0.45 - 0.1 = 0.35$.

(We never use $P(C) = 0.25$.)

5. The following tree shows the setting

Let $C$ be the event that you answer the question correctly. Let $K$ be the event that you actually know the answer. The left circled node shows $P(K \cap C) = p$. Both circled nodes together show $P(C) = p + (1 - p)/c$. So,

\[
P(K|C) = \frac{P(K \cap C)}{P(C)} = \frac{p}{p + (1 - p)/c}
\]

Or we could use the algebraic form of Bayes theorem and the law of total probability: Let $G$ stand for the event that you’re guessing. Then we have,

$P(C|K) = 1$, $P(K) = p$, $P(C) = P(C|K)P(K) + P(C|G)P(G) = p + (1 - p)/c$. So,

\[
P(K|C) = \frac{P(C|K)P(K)}{P(C)} = \frac{p}{p + (1 - p)/c}
\]

6. Sample space =

$\Omega = \{(1,1), (1,2), (1,3), \ldots , (6,6)\} = \{(i,j)| i,j = 1, 2, 3, 4, 5, 6\}$.

(Each outcome is equally likely, with probability $1/36$.)

$A = \{(1,2), (2,1)\}$,

$B = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$

$C = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (3,1), (4,1), (5,1), (6,1)\}$

(a) $P(A|C) = \frac{P(A \cap C)}{P(C)} = \frac{2/36}{11/36} = \frac{2}{11}$. 
(a) \( P(B|C) = \frac{P(B \cap C)}{P(C)} = \frac{2/36}{11/36} = \frac{2}{11} \).

(c) \( P(A) = 2/36 \neq P(A|C) \), so they are not independent. Similarly, \( P(B) = 6/36 \neq P(B|C) \), so they are not independent.

7. We show the probabilities in a tree:

\[
\begin{array}{c}
\text{Know} \quad \frac{1}{2} \quad \frac{1}{4} \quad \frac{1}{4} \\
\text{Correct} \quad 0 \quad 1/3 \quad 2/3 \quad 1/4 \quad 3/4 \\
\text{Wrong} \quad \text{Correct} \quad \text{Wrong} \quad \text{Correct} \quad \text{Wrong}
\end{array}
\]

For a given problem let \( C \) be the event the student gets the problem correct and \( K \) the event the student knows the answer.

The question asks for \( P(K|C) \).

We'll compute this using Bayes’ rule:

\[
P(K|C) = \frac{P(C|K) P(K)}{P(C)} = \frac{1 \cdot 1/2}{1/2 + 1/12 + 1/16} = \frac{24}{31} \approx 0.774 = 77.4%\]

8. We have \( P(A \cup B) = 1 - 0.42 = 0.58 \) and we know because of the inclusion-exclusion principle that

\[P(A \cup B) = P(A) + P(B) - P(A \cap B).\]

Thus,

\[P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.4 + 0.3 - 0.58 = 0.12 = (0.4)(0.3) = P(A)P(B)\]

So \( A \) and \( B \) are independent.

9. We will make use of the formula \( \text{Var}(Y) = E(Y^2) - E(Y)^2 \). First we compute

\[E[X] = \int_0^1 x \cdot 2x \, dx = \frac{2}{3}\]

\[E[X^2] = \int_0^1 x^2 \cdot 2x \, dx = \frac{1}{2}\]

\[E[X^4] = \int_0^1 x^4 \cdot 2x \, dx = \frac{1}{3}\].

Thus,

\[\text{Var}(X) = E[X^2] - (E[X])^2 = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}\]

and

\[\text{Var}(X^2) = E[X^4] - (E[X^2])^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}\].
10. Make a table

<table>
<thead>
<tr>
<th>$X$</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>prob</td>
<td>$(1-p)$</td>
<td>$p$</td>
</tr>
<tr>
<td>$X^2$</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

From the table, $E(X) = 0 \cdot (1 - p) + 1 \cdot p = p$.

Since $X$ and $X^2$ have the same table $E(X^2) = E(X) = p$.

Therefore, $\text{Var}(X) = p - p^2 = p(1 - p)$.

11. Let $X$ be the number of people who get their own hat.

Following the hint: let $X_j$ represent whether person $j$ gets their own hat. That is, $X_j = 1$ if person $j$ gets their hat and 0 if not.

We have, $X = \sum_{j=1}^{100} X_j$, so $E(X) = \sum_{j=1}^{100} E(X_j)$.

Since person $j$ is equally likely to get any hat, we have $P(X_j = 1) = 1/100$. Thus, $X_j \sim \text{Bernoulli}(1/100) \Rightarrow E(X_j) = 1/100 \Rightarrow E(X) = 1$.

12. For $y = 0, 2, 4, \ldots, 2n$,

$$P(Y = y) = P(X = \frac{y}{2}) = \left(\frac{n}{y/2}\right) \left(\frac{1}{2}\right)^n.$$ 

13. The CDF for $R$ is

$$F_R(r) = P(R \leq r) = \int_0^r 2e^{-2u} \, du = -e^{-2u}\big|_0^r = 1 - e^{-2r}.$$ 

Next, we find the CDF of $T$. $T$ takes values in $(0, \infty)$.

For $0 < t$,

$$F_T(t) = P(T \leq t) = P(1/R < t) = P(1/t > R) = 1 - F_R(1/t) = e^{-2/t}.$$ 

We differentiate to get $f_T(t) = \frac{d}{dt} \left(e^{-2/t}\right) = \frac{2}{t^2}e^{-2/t}$

14. The jumps in the distribution function are at 0, 2, 4. The value of $p(a)$ at a jump is the height of the jump:

<table>
<thead>
<tr>
<th>$a$</th>
<th>$p(a)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1/5</td>
</tr>
<tr>
<td>2</td>
<td>1/5</td>
</tr>
<tr>
<td>4</td>
<td>3/5</td>
</tr>
</tbody>
</table>

15. We compute

$$P(X \geq 5) = 1 - P(X < 5) = 1 - \int_0^5 \lambda e^{-\lambda x} \, dx = 1 - (1 - e^{-5\lambda}) = e^{-5\lambda}.$$ 

(b) We want $P(X \geq 15|X \geq 10)$. First observe that $P(X \geq 15, X \geq 10) = P(X \geq 15)$.

From similar computations in (a), we know

$$P(X \geq 15) = e^{-15\lambda} \quad P(X \geq 10) = e^{-10\lambda}.$$
From the definition of conditional probability,
\[ P(X \geq 15|X \geq 10) = \frac{P(X \geq 15, X \geq 10)}{P(X \geq 10)} = \frac{P(X \geq 15)}{P(X \geq 10)} = e^{-5\lambda} \]

Note: This is an illustration of the memorylessness property of the exponential distribution.

16. Transforming Normal Distributions
(a) Note, \( Y \) follows what is called a log-normal distribution.
\[ F_Y(a) = P(Y \leq a) = P(e^Z \leq a) = P(Z \leq \ln(a)) = \Phi(\ln(a)). \]
Differentiating using the chain rule:
\[ f_y(a) = \frac{d}{da} F_Y(a) = \frac{d}{da} \Phi(\ln(a)) = \frac{1}{\sqrt{2\pi}a} e^{-\ln(a)^2/2}. \]

(b) (i) The 0.33 quantile for \( Z \) is the value \( q_{0.33} \) such that \( P(Z \leq q_{0.33}) = 0.33 \). That is, we want
\[ \Phi(q_{0.33}) = 0.33 \iff q_{0.33} = \Phi^{-1}(0.33). \]
(ii) We want to find \( q_{0.9} \) where
\[ F_Y(q_{0.9}) = 0.9 \iff \Phi(\ln(q_{0.9})) = 0.9 \iff q_{0.9} = e^{\Phi^{-1}(0.9)}. \]
(iii) As in (ii) \( q_{0.5} = e^{\Phi^{-1}(0.5)} = e^0 = 1 \).

17. (a) Total probability must be 1, so
\[ 1 = \int_0^3 \int_0^3 f(x,y) \, dy \, dx = \int_0^3 \int_0^3 c(x^2y + x^2y^2) \, dy \, dx = c \cdot \frac{243}{2}, \]
(Here we skipped showing the arithmetic of the integration) Therefore, \( c = \frac{2}{243} \).
(b)
\[ P(1 \leq X \leq 2, 0 \leq Y \leq 1) = \int_1^2 \int_0^1 f(x,y) \, dy \, dx = \int_1^2 \int_0^1 c(x^2y + x^2y^2) \, dy \, dx = c \cdot \frac{35}{18} = \frac{70}{4374} \approx 0.016 \]
(c) For \( 0 \leq a \leq 1 \) and \( 0 \leq b \leq 1 \), we have
\[ F(a,b) = \int_0^a \int_0^b f(x,y) \, dy \, dx = c \left( \frac{a^3b^2}{6} + \frac{a^3b^3}{9} \right) \]
(d) Since \( y = 3 \) is the maximum value for \( Y \), we have
\[
F_X(a) = F(a, 3) = c \left( \frac{9a^3}{6} + 3a^3 \right) = \frac{9}{2}ca^3 = \frac{a^3}{27}
\]

(e) For \( 0 \leq x \leq 3 \), we have, by integrating over the entire range for \( y \),
\[
f_X(x) = \int_0^3 f(x, y) \, dy = cx^2 \left( \frac{3^2}{2} + \frac{3^3}{3} \right) = \frac{27}{2}x^2 = \frac{1}{9}x^2.
\]
This is consistent with (c) because \( \frac{d}{dx}(x^3/27) = x^2/9 \).

(f) Since \( f(x, y) \) separates into a product as a function of \( x \) times a function of \( y \) we know \( X \) and \( Y \) are independent.

18. (Central Limit Theorem) Let \( T = X_1 + X_2 + \ldots + X_{81} \). The central limit theorem says that
\[
T \approx N(81 \times 5, 81 \times 4) = N(405, 18^2)
\]

Standardizing we have
\[
P(T > 369) = P \left( \frac{T - 405}{18} > \frac{369 - 405}{18} \right) \\
\approx P(Z > -2) \\
\approx 0.975
\]

The value of 0.975 comes from the rule-of-thumb that \( P(|Z| < 2) \approx 0.95 \). A more exact value (using R) is \( P(Z > -2) \approx 0.9772 \).