1. (12 points) This question is about the matrix

\[
A = \begin{bmatrix}
1 & 2 & 0 & 1 \\
2 & 4 & 1 & 4 \\
3 & 6 & 3 & 9
\end{bmatrix}.
\]

(a) Find a lower triangular \( L \) and an upper triangular \( U \) so that \( A = LU \).

(b) Find the reduced row echelon form \( R = \text{rref}(A) \). How many independent columns in \( A \)?

(c) Find a basis for the nullspace of \( A \).

(d) If the vector \( b \) is the sum of the four columns of \( A \), write down the complete solution to \( Ax = b \).
2. (11 points) This problem finds the curve $y = C + D 2^t$ which gives the best least squares fit to the points $(t, y) = (0, 6), (1, 4), (2, 0)$.

(a) Write down the 3 equations that would be satisfied if the curve went through all 3 points.

(b) Find the coefficients $C$ and $D$ of the best curve $y = C + D 2^t$.

(c) What values should $y$ have at times $t = 0, 1, 2$ so that the best curve is $y = 0$?
3. **(11 points)** Suppose $Av_i = b_i$ for the vectors $v_1, \ldots, v_n$ and $b_1, \ldots, b_n$ in $\mathbb{R}^n$. Put the $v$’s into the columns of $V$ and put the $b$’s into the columns of $B$.

(a) Write those equations $Av_i = b_i$ in matrix form. What condition on which vectors allows $A$ to be determined uniquely? Assuming this condition, find $A$ from $V$ and $B$.

(b) Describe the column space of that matrix $A$ in terms of the given vectors.

(c) What additional condition on which vectors makes $A$ an invertible matrix? Assuming this, find $A^{-1}$ from $V$ and $B$. 
4. (11 points)

(a) Suppose $x_k$ is the fraction of MIT students who prefer calculus to linear algebra at year $k$. The remaining fraction $y_k = 1 - x_k$ prefers linear algebra.

At year $k + 1$, $1/5$ of those who prefer calculus change their mind (possibly after taking 18.03). Also at year $k + 1$, $1/10$ of those who prefer linear algebra change their mind (possibly because of this exam).

Create the matrix $A$ to give

$$
\begin{bmatrix}
x_{k+1} \\
y_{k+1}
\end{bmatrix} = A
\begin{bmatrix}
x_k \\
y_k
\end{bmatrix}
$$

and find the limit of $A^k
\begin{bmatrix}
1 \\
0
\end{bmatrix}$ as $k \to \infty$.

(b) Solve these differential equations, starting from $x(0) = 1$, $y(0) = 0$:

$$
\frac{dx}{dt} = 3x - 4y, \quad \frac{dy}{dt} = 2x - 3y.
$$

(c) For what initial conditions $\begin{bmatrix}
x(0) \\
y(0)
\end{bmatrix}$ does the solution $\begin{bmatrix}
x(t) \\
y(t)
\end{bmatrix}$ to this differential equation lie on a single straight line in $R^2$ for all $t$?
5. (11 points)

(a) Consider a 120° rotation around the axis $x = y = z$. Show that the vector $i = (1, 0, 0)$ is rotated to the vector $j = (0, 1, 0)$. (Similarly $j$ is rotated to $k = (0, 0, 1)$ and $k$ is rotated to $i$.) How is $j - i$ related to the vector $(1, 1, 1)$ along the axis?

(b) Find the matrix $A$ that produces this rotation (so $Av$ is the rotation of $v$). Explain why $A^3 = I$. What are the eigenvalues of $A$?

(c) If a 3 by 3 matrix $P$ projects every vector onto the plane $x + 2y + z = 0$, find three eigenvalues and three independent eigenvectors of $P$. No need to compute $P$. 
6. (11 points) This problem is about the matrix

\[ A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}. \]

(a) Find the eigenvalues of \( A^T A \) and also of \( AA^T \). For both matrices find a complete set of orthonormal eigenvectors.

(b) If you apply the Gram-Schmidt process (orthonormalization) to the columns of this matrix \( A \), what is the resulting output?

(c) If \( A \) is any \( m \) by \( n \) matrix with \( m > n \), tell me why \( AA^T \) cannot be positive definite. Is \( A^T A \) always positive definite? (If not, what is the test on \( A \)?)
7. **(11 points)** This problem is to find the determinants of

\[
A = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 \\
1 & 1 & 0 & 0
\end{bmatrix}, \\
B = \begin{bmatrix}
0 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 \\
1 & 1 & 0 & 0
\end{bmatrix}, \\
C = \begin{bmatrix}
x & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 \\
1 & 1 & 0 & 0
\end{bmatrix}
\]

(a) Find \(\det A\) and give a reason.

(b) Find the cofactor \(C_{11}\) and then find \(\det B\). This is the volume of what region in \(\mathbb{R}^4\)?

(c) Find \(\det C\) for any value of \(x\). You could use linearity in row 1.
8. (11 points)

(a) When $A$ is similar to $B = M^{-1}AM$, prove this statement:
   If $A^k \to 0$ when $k \to \infty$, then also $B^k \to 0$.

(b) Suppose $S$ is a fixed invertible 3 by 3 matrix.
   This question is about all the matrices $A$ that are diagonalized by $S$, so that
   $S^{-1}AS$ is diagonal. Show that these matrices $A$ form a subspace of
   3 by 3 matrix space. (Test the requirements for a subspace.)

(c) Give a basis for the space of 3 by 3 diagonal matrices. Find a basis for the space in part (b)
   — all the matrices $A$ that are diagonalized by $S$. 
9. **(11 points)** This square network has 4 nodes and 6 edges. On each edge, the direction of positive current $w_i > 0$ is from lower node number to higher node number. The voltages at the nodes are $(v_1, v_2, v_3, v_4)$.

(a) Write down the incidence matrix $A$ for this network (so that $Av$ gives the 6 voltage differences like $v_2 - v_1$ across the 6 edges). What is the rank of $A$? What is the dimension of the nullspace of $A^T$?

(b) Compute the matrix $A^T A$. What is its rank? What is its nullspace?

(c) Suppose $v_1 = 1$ and $v_4 = 0$. If each edge contains a unit resistor, the currents $(w_1, w_2, w_3, w_4, w_5, w_6)$ on the 6 edges will be $w = -Av$ by Ohm’s Law. Then Kirchhoff’s Current Law (flow in = flow out at every node) gives $A^T w = 0$ which means $A^T Av = 0$. Solve $A^T Av = 0$ for the unknown voltages $v_2$ and $v_3$. Find all 6 currents $w_1$ to $w_6$. How much current enters node 4?