1. Forward elimination changes \( Ax = b \) to a row reduced \( Rx = d \): the complete solution is

\[
x = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}
\]

(a) \((14 \text{ points})\) What is the 3 by 3 reduced row echelon matrix \( R \) and what is \( d \)?

Solution: First, since \( R \) is in reduced row echelon form, we must have

\[
d = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}^T
\]

The other two vectors provide special solutions for \( R \), showing that \( R \) has rank 1: again, since it is in reduced row echelon form, the bottom two rows must be all 0, and

the top row is \( \begin{bmatrix} 1 & -2 & -5 \end{bmatrix}^T \), i.e.

\[
R = \begin{bmatrix} 1 & -2 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

(b) \((10 \text{ points})\) If the process of elimination subtracted 3 times row 1 from row 2 and then 5 times row 1 from row 3, what matrix connects \( R \) and \( d \) to the original \( A \) and \( b \)? Use this matrix to find \( A \) and \( b \).

Solution: The matrix connecting \( R \) and \( d \) to the original \( A \) and \( b \) is

\[
E = E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ -3 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \\ -5 & 0 \\ 1 & 0 \end{bmatrix}
\]

That is, \( R = EA \) and \( Eb = d \). Thus, \( A = E^{-1}R \) and \( b = E^{-1}d \), giving
2. Suppose $A$ is the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & -5 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -5 \\ 3 & -6 & -15 \\ 5 & -10 & -25 \end{bmatrix}$$

$$b = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \\ 20 \end{bmatrix}$$

(a) **(16 points)** Find all special solutions to $Ax = 0$ and describe in words the whole nullspace of $A$.

Solution: First, by row reduction

$$\begin{bmatrix} 0 & 1 & 2 & 2 \\ 0 & 3 & 8 & 7 \\ 0 & 0 & 4 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 4 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so the special solutions are

$$s_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad s_2 = \begin{bmatrix} 0 \\ -1 \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

Thus, $N(A)$ is a plane in $\mathbb{R}^4$ given by all linear combinations of the special solutions.

(b) **(10 points)** Describe the column space of this particular matrix $A$. “All combinations of the four columns” is not a sufficient answer.

Solution: $C(A)$ is a plane in $\mathbb{R}^3$ given by all combinations of the pivot columns, namely

$$c_1 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 8 \\ 4 \end{bmatrix}$$
(c) **(10 points)** What is the reduced row echelon form $R^* = \text{rref}(B)$ when $B$ is the 6 by 8 block matrix

\[
B = \begin{bmatrix}
A & A \\
A & A
\end{bmatrix}
\]

using the same $A$?

**Solution:** Note that $B$ immediately reduces to

\[
B = \begin{bmatrix}
A & A \\
0 & 0
\end{bmatrix}
\]

We reduced $A$ above: the row reduced echelon form of $B$ is thus

\[
B = \begin{bmatrix}
\text{rref}(A) & \text{rref}(A) \\
0 & 0
\end{bmatrix}, \text{rref}(A) = \begin{bmatrix} 0 & 1 & 0 & 1 \\
0 & 0 & 1 & \frac{1}{2} \\
0 & 0 & 0 & 0 \end{bmatrix}
\]

3. **(16 points)** Circle the words that correctly complete the following sentence:

(a) Suppose a 3 by 5 matrix $A$ has rank $r = 3$. Then the equation $Ax = b$

\[
( \text{ always / sometimes but not always } )
\]

has ( a unique solution / many solutions / no solution ).

**Solution:** the equation $Ax = b$ **always** has **many solutions**.

(b) What is the column space of $A$? Describe the nullspace of $A$.

**Solution:** The column space is a 3-dimensional space inside a 3-dimensional space, i.e.

it contains all the vectors, and **the nullspace has dimension $5 - 3 = 2 > 0$ inside $\mathbb{R}^5$.**
4. Suppose that $A$ is the matrix 

$$
A = \begin{bmatrix}
2 & 1 \\
6 & 5 \\
2 & 4 \\
\end{bmatrix}.
$$

(a) (10 points) Explain in words how knowing all solutions to $Ax = b$ decides if a given vector $b$ is in the column space of $A$.

Solution: The column space of $A$ contains all linear combinations of the columns of $A$, which are precisely vectors of the form $Ax$ for an arbitrary vector $x$. Thus, $Ax = b$ has a solution if and only if $b$ is in the column space of $A$.

(b) (14 points) Is the vector $b = \begin{bmatrix} 8 \\ 28 \\ 14 \end{bmatrix}$ in the column space of $A$?

Solution: Yes. Reducing the matrix combining $A$ and $b$ gives

$$
\begin{bmatrix}
2 & 1 & 8 \\
6 & 5 & 28 \\
2 & 4 & 14 \\
\end{bmatrix} \rightarrow
\begin{bmatrix}
2 & 1 & 8 \\
0 & 2 & 4 \\
0 & 3 & 6 \\
\end{bmatrix} \rightarrow
\begin{bmatrix}
2 & 1 & 8 \\
0 & 2 & 4 \\
0 & 0 & 0 \\
\end{bmatrix}
$$

Thus, $x = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ is a solution to $Ax = b$, and $b$ is in the column space of $A$. 
