PROBLEM 1.

The field \( \mathbb{Q} \) is a linear space over \( \mathbb{Q} \), but not over \( \mathbb{R} \). In order to prove the latter statement by contradiction, suppose that \( \mathbb{Q} \) is a linear space over \( \mathbb{R} \). For \( v \neq 0, v \in \mathbb{Q}, f, g \in \mathbb{R}, f v = g v \Rightarrow f = g \) and \( f v \neq g v \Rightarrow f \neq g \). This then would give a one-to-one map from \( \mathbb{R} \) to \( \mathbb{Q} \) defined as \( f \mapsto f v \). However, since the cardinality of \( \mathbb{R} \) is greater than the cardinality of \( \mathbb{Q} \), there cannot be a one-to-one map from \( \mathbb{R} \) to \( \mathbb{Q} \). Thus, by contradiction, \( \mathbb{Q} \) is not a linear space over \( \mathbb{R} \). Conversely, one-to-one map from \( \mathbb{Q} \) to \( \mathbb{R} \) can be defined as \( q \mapsto qr, q \in \mathbb{Q}, r \in \mathbb{R} \). Hence, by restricting the coefficients from \( \mathbb{R} \) to \( \mathbb{Q} \), any linear space over \( \mathbb{R} \) becomes a linear space over \( \mathbb{Q} \).

PROBLEM 2.

(a) Yes, sequences with only finitely many nonzero elements are a subspace of \( A \). Let \( S \) be all the infinite sequences over \( \mathbb{R} \) with finitely many non-zero terms and let \( a, b \in S, k \in \mathbb{R} \). It is clear that \( a + kb \in S \) since the number of non-zero terms will still be finite.

(b) No, sequences with only finitely many zero terms are not a subspace of \( A \). Let \( S \) be all the infinite sequences over \( \mathbb{R} \) with only finitely many zero terms and let \( a \in S \). Since \( 0 \cdot a = 0 \notin S \), \( S \) is not a linear space.

(c) Yes, Cauchy sequences are a subspace of \( A \). Let \( S \) be the set of all Cauchy sequences and \( a, b \in S \). Suppose \( \varepsilon_{ab} \) is given and choose \( \varepsilon_a, \varepsilon_b > 0 \) such that \( \varepsilon_{ab} = \varepsilon_a + \varepsilon_b \). Find \( N_a, N_b \in \mathbb{R} \) such that \(|a_n - a_m| < \varepsilon_a\) for all \( m, n > N_a \) (similarly for \( b \)). We need to locate \( N_{ab} \) such that \(|(a_n + b_n) - (a_m + b_m)| < \varepsilon_{ab}\) for all \( m, n > N_{ab} \). From triangle inequality \(|A + B| \leq |A| + |B|\). Hence, for \( N_{ab} = \max(N_a, N_b), |(a_n - a_m) + (b_n - b_m)| \leq \varepsilon_a + \varepsilon_b = \varepsilon_{ab} \).

(d) Yes, the sequences, for which the sum of the squares of the elements converges, is a subspace of \( A \). Let \( S \) be the set of all the infinite sequences \( \{a_i\}_{i=1}^{\infty}, a_i \in \mathbb{R} \) for which \( \sum_{i=1}^{\infty} a_i^2 \) converges. Then for \( a, b \in S, a + b \in S \): \( \sum (a_i + b_i)^2 = \sum a_i^2 + \sum b_i^2 + 2 \sum a_i b_i \). By Cauchy-Schwarz \( (\sum x_i^2) \cdot (\sum y_i^2) \geq (\sum a_i b_i)^2 \). Also, for \( k \in \mathbb{R}, ka \in S \): \( \sum (ka_i)^2 = k^2 \cdot \sum a_i^2 \). Therefore, \( S \) is a linear space.

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