ABSTRACT ROOT SYSTEMS

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Let $V$ be a Euclidean space, that is, a real, finite-dimensional vector space with a symmetric, positive-definite inner product $\langle , \rangle$. Recall the definition of a reflection in $V$ from [1]:

**Definition 1.** A reflection of a vector $\vec{x} \in V$ with respect to a vector $\vec{\alpha} \in V$ is defined by the formula

$$s_{\vec{\alpha}}(\vec{x}) = \vec{x} - \frac{2\langle \vec{x}, \vec{\alpha} \rangle}{\langle \vec{\alpha}, \vec{\alpha} \rangle} \vec{\alpha}.$$

We can now define an abstract root system in a Euclidean space.

**Definition 2.** An abstract root system in $V$ is a finite set $\Delta$ of nonzero elements of $V$ such that

1. $\Delta$ spans $V$;
2. for all $\vec{\alpha} \in \Delta$, the reflections

$$s_{\vec{\alpha}}(\vec{\beta}) = \vec{\beta} - \frac{2\langle \vec{\beta}, \vec{\alpha} \rangle}{\langle \vec{\alpha}, \vec{\alpha} \rangle} \vec{\alpha}$$

map the set $\Delta$ to itself;
3. the number $\frac{2\langle \vec{\beta}, \vec{\alpha} \rangle}{\langle \vec{\alpha}, \vec{\alpha} \rangle}$ is an integer for any $\vec{\alpha}, \vec{\beta} \in \Delta$.

A root is an element of $\Delta$.

We will begin by considering some examples of root systems.

**Example 3.** Let $V$ be the following subspace of $\mathbb{R}^{n+1}$, $n \geq 1$:

$$V = \left\{ \sum_{i=1}^{n+1} a_i \vec{e}_i, \text{ with } \sum_{i=1}^{n+1} a_i = 0 \right\},$$

where $\{ \vec{e}_i \}_{i=1}^{n+1}$ is an orthonormal basis in $\mathbb{R}^{n+1}$, and all $a_i \in \mathbb{R}$.

**Claim.** The set $\Delta = \{ \vec{e}_i - \vec{e}_j, i \neq j \}$ is an abstract root system.

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Proof. We must first show that $\Delta$ spans $V$. Construct $\tilde{\Delta} \subset \Delta$ where

$$\tilde{\Delta} = \{\tilde{e}_2 - \tilde{e}_1, \tilde{e}_3 - \tilde{e}_1, \tilde{e}_4 - \tilde{e}_1, \ldots, \tilde{e}_n - \tilde{e}_1, \tilde{e}_{n+1} - \tilde{e}_1\}.$$  

If we show that $\tilde{\Delta}$ spans $V$, then $\Delta$ necessarily spans $V$ as well.

A vector $\vec{v} \in V$ can be written as

$$\vec{v} = a_1\tilde{e}_1 + a_2\tilde{e}_2 + \cdots + a_n\tilde{e}_n + a_{n+1}\tilde{e}_{n+1} \tag{2}$$

where

$$a_1 + a_2 + \cdots + a_n + a_{n+1} = 0. \tag{3}$$

Rewrite (3) as

$$a_1 = -(a_2 + a_3 + \cdots + a_n + a_{n+1}) \tag{4}$$

and substitute (4) into (2) to get

$$\vec{v} = -(a_2 + a_3 + \cdots + a_n + a_{n+1})\tilde{e}_1 + a_2\tilde{e}_2 + \cdots + a_n\tilde{e}_n + a_{n+1}\tilde{e}_{n+1}. \tag{5}$$

We can then simplify (5):

$$\vec{v} = a_2(\tilde{e}_2 - \tilde{e}_1) + a_3(\tilde{e}_3 - \tilde{e}_1) + \cdots + a_n(\tilde{e}_n - \tilde{e}_1) + a_{n+1}(\tilde{e}_{n+1} - \tilde{e}_1). \tag{6}$$

Equation (6) clearly shows that any $\vec{v} \in V$ can be written as a linear combination of the elements of $\tilde{\Delta}$. Hence, $\tilde{\Delta}$ spans $V$, and therefore $\Delta$ spans $V$.

Next, we must show that for any $\alpha, \beta \in \Delta$, the reflections $s_\alpha(\beta)$ map the set $\Delta$ to itself. Take $\alpha = \tilde{e}_i - \tilde{e}_j$ and $\beta = \tilde{e}_k - \tilde{e}_m$, where $i \neq j$ and $k \neq m$.

Apply the reflection:

$$s_{\tilde{e}_i - \tilde{e}_j}(\tilde{e}_k - \tilde{e}_m) = \tilde{e}_k - \tilde{e}_m - \frac{2(\tilde{e}_k - \tilde{e}_m, \tilde{e}_i - \tilde{e}_j)}{(\tilde{e}_i - \tilde{e}_j, \tilde{e}_i - \tilde{e}_j)}(\tilde{e}_i - \tilde{e}_j) \tag{7}$$

By the symmetry and bilinearity of the inner product, we can simplify (7) as follows:

$$s_{\tilde{e}_i - \tilde{e}_j}(\tilde{e}_k - \tilde{e}_m) = \tilde{e}_k - \tilde{e}_m - \frac{2(\tilde{e}_k - \tilde{e}_m, \tilde{e}_i - \tilde{e}_j)}{(\tilde{e}_i - \tilde{e}_j, \tilde{e}_i - \tilde{e}_j)}(\tilde{e}_i - \tilde{e}_j). \tag{8}$$

Since $\{\tilde{e}_i\}_{i=1}^{n+1}$ is an orthonormal basis, we know that $\langle \tilde{e}_i, \tilde{e}_i \rangle = 1$ and $\langle \tilde{e}_i, \tilde{e}_j \rangle = 0$ if $i \neq j$. We can simplify the fraction in (8). The denominator is clearly 2, which cancels the 2 in the numerator. Hence,

$$s_{\tilde{e}_i - \tilde{e}_j}(\tilde{e}_k - \tilde{e}_m) = \tilde{e}_k - \tilde{e}_m - [(\tilde{e}_k, \tilde{e}_i) - (\tilde{e}_m, \tilde{e}_i)](\tilde{e}_i - \tilde{e}_j). \tag{9}$$

It therefore follows that

$$s_{\tilde{e}_i - \tilde{e}_j}(\tilde{e}_k - \tilde{e}_m) = \begin{cases} 
\tilde{e}_m - \tilde{e}_k & \text{if } i = k, j = m \text{ or } i = m, j = k \\
\tilde{e}_k - \tilde{e}_m & \text{if } i \neq k \neq j \neq m \neq i \\
\tilde{e}_j - \tilde{e}_m & \text{if } i = k, j \neq m \\
\tilde{e}_k - \tilde{e}_j & \text{if } i = m, j \neq k \\
\tilde{e}_i - \tilde{e}_m & \text{if } i \neq m, j = k \\
\tilde{e}_k - \tilde{e}_i & \text{if } j = m, i \neq k 
\end{cases} \tag{10}$$
All possible cases in (10) are the difference of two distinct elements of the orthonormal basis, which is exactly the definition of elements of $\Delta$, so $\Delta$ is invariant under reflection and the second property is satisfied. In addition, for all cases the denominator of the fraction in (8) is 2, exactly canceling the 2 in the numerator. The sum of the inner products in the numerator is a combination of 0s and 1s, so it is always an integer. Thus, the fraction

$$\frac{2\langle \vec{\alpha}, \vec{\beta} \rangle}{\langle \vec{\alpha}, \vec{\alpha} \rangle} = \frac{2(\langle e_k, e_i \rangle - \langle e_i, e_j \rangle - \langle e_m, e_i \rangle + \langle e_m, e_j \rangle)}{\langle e_i, e_i \rangle - \langle e_i, e_j \rangle - \langle e_j, e_i \rangle + \langle e_j, e_j \rangle}$$

is always an integer, which satisfies the third condition. \( \square \)

Root systems as defined in example 3 are of the type $A_n$. We will now consider the geometry of $A_1$ and $A_2$.

For $n = 1$, $V$ is the subspace of $\mathbb{R}^2$ where $V = \{a_1(\vec{e}_1 - \vec{e}_2) | a_1 \in \mathbb{R}\}$. Thus, $\Delta = \{\vec{e}_1 - \vec{e}_2, \vec{e}_2 - \vec{e}_1\}$. The standard basis is orthonormal, so we can write $\vec{e}_1 = (1,0)$ and $\vec{e}_2 = (0,1)$. As a result, $\Delta = \{(1,-1), (-1,1)\}$ as depicted in figure 1.

**Figure 1.** $A_1$ Abstract Root System

![Figure 1](attachment:image1.png)

If $n = 2$, $V = \{a_1\vec{e}_1 + a_2\vec{e}_2 + a_3\vec{e}_3 | a_1 + a_2 + a_3 = 0\}$ is a subspace of $\mathbb{R}^3$. It is clear that $\Delta = \{\vec{e}_1 - \vec{e}_2, \vec{e}_1 - \vec{e}_3, \vec{e}_2 - \vec{e}_3, \vec{e}_2 - \vec{e}_1, \vec{e}_3 - \vec{e}_1, \vec{e}_3 - \vec{e}_2\}$. Using the standard basis in three-space, $\vec{e}_1 = (1,0,0)$, $\vec{e}_2 = (0,1,0)$, and $\vec{e}_3 = (0,0,1)$, we observe that

$$\Delta = \{(1,-1,0),(1,0,-1),(0,1,1),(-1,1,0),(-1,0,1),(0,-1,1)\},$$

which is shown geometrically in figure 2. If we connect the roots of $A_2$, we see that we get a regular hexagon of side length $\sqrt{2}$, as shown in figure 3.

Use of the standard basis is the simplest way to visualize $A_1$ and $A_2$, yet any orthonormal basis could have been chosen in each case.
Example 4. Let $V$ be the space $\mathbb{R}^n$ such that $n \geq 2$ with an orthonormal basis $\{\vec{e}_i\}_{i=1}^n$.

Claim. The set $\Delta = \{\pm \vec{e}_i \pm \vec{e}_j, i \neq j\} \cup \{\pm \vec{e}_i\}$ is an abstract root system.

Proof. The set $\Delta$ clearly spans $V$ since it contains the orthonormal basis $\{\vec{e}_i\}_{i=1}^n$. We must next show that for all $\vec{\alpha}, \vec{\beta} \in \Delta$, the reflection $s_{\vec{\alpha}}(\vec{\beta}) \in \Delta$. Define $\{\vec{b}_i\}_{i=1}^n = \{\vec{e}_i\}_{i=1}^n$ where all $\vec{e}_i = \vec{b}_i$ and let $\vec{b}_{n+1} = \vec{0}$. We can now write $\Delta$ as $\Delta = \{\pm \vec{b}_i \pm \vec{b}_j, i \neq j\}$.

Take $\vec{\alpha} = \pm \vec{b}_i \pm \vec{b}_j$ and $\vec{\beta} = \pm \vec{b}_k \pm \vec{b}_m$, such that $i \neq j$ and $k \neq m$. We must show that $s_{\vec{\alpha}}(\vec{\beta}) \in \Delta$ for all $\vec{\alpha}, \vec{\beta} \in \Delta$. Begin by expanding the equation for the reflection:

\begin{equation}
\begin{aligned}
    s_{\pm \vec{b}_i \pm \vec{b}_j}(\pm \vec{b}_k \pm \vec{b}_m) &= \pm \vec{b}_k \pm \vec{b}_m - 2\frac{\langle \pm \vec{b}_k \pm \vec{b}_m, \pm \vec{b}_i \pm \vec{b}_j \rangle}{\langle \pm \vec{b}_i \pm \vec{b}_j, \pm \vec{b}_i \pm \vec{b}_j \rangle}(\pm \vec{b}_i \pm \vec{b}_j).
\end{aligned}
\end{equation}
We will first consider the fraction in (11). By the bilinearity of the inner product and by the orthonormality of the $\tilde{b}_i$, we can simplify the denominator as follows:
\[
\langle \pm \tilde{b}_i \pm \tilde{b}_j, \pm \tilde{b}_i \pm \tilde{b}_j \rangle = \langle \pm \tilde{b}_i, \pm \tilde{b}_i \rangle + \langle \pm \tilde{b}_j, \pm \tilde{b}_j \rangle + \langle \pm \tilde{b}_i, \pm \tilde{b}_j \rangle + \langle \pm \tilde{b}_j, \pm \tilde{b}_i \rangle = 1 + 0 + 0 + 1 = 2.
\]
This 2 in the denominator cancels the 2 in the numerator. We are now left with
\[
\langle \pm \tilde{b}_k, \pm \tilde{b}_m, \pm \tilde{b}_i \pm \tilde{b}_j \rangle = \langle \pm \tilde{b}_k, \pm \tilde{b}_k \rangle + \langle \pm \tilde{b}_k, \pm \tilde{b}_j \rangle + \langle \pm \tilde{b}_m, \pm \tilde{b}_i \rangle + \langle \pm \tilde{b}_m, \pm \tilde{b}_j \rangle
\]
Finally, simplify the remaining inner product.
\[
\langle \pm \tilde{b}_k, \pm \tilde{b}_m, \pm \tilde{b}_i \pm \tilde{b}_j \rangle = \langle \pm \tilde{b}_k, \pm \tilde{b}_i \rangle + \langle \pm \tilde{b}_k, \pm \tilde{b}_j \rangle + \langle \pm \tilde{b}_m, \pm \tilde{b}_i \rangle + \langle \pm \tilde{b}_m, \pm \tilde{b}_j \rangle
\]
It therefore follows that
\[
s_{\pm \tilde{b}_i \pm \tilde{b}_j}(\pm \tilde{b}_k \pm \tilde{b}_m) = \begin{cases} 
\pm \tilde{b}_k \pm \tilde{b}_j & \text{if } i = k, j = m \text{ or } i = m, j = k \\
\pm \tilde{b}_k \pm \tilde{b}_m & \text{if } i \neq k \neq j \neq m \neq i \\
\pm \tilde{b}_m \pm \tilde{b}_j & \text{if } i = k, j \neq m \\
\pm \tilde{b}_k \pm \tilde{b}_j & \text{if } i = m, j \neq k \\
\pm \tilde{b}_m \pm \tilde{b}_j & \text{if } i \neq m, j = k \\
\pm \tilde{b}_k \pm \tilde{b}_i & \text{if } j = m, i \neq k 
\end{cases}
\]
All 6 cases in (14) are elements of $\Delta$, so the second property of an abstract root system is satisfied. Furthermore, all the possible cases in (14) indicate that the inner product in (13) can only be 0, ±1, or ±2. Hence, the fraction
\[
\frac{2\langle \bar{\alpha}, \bar{\beta} \rangle}{\langle \bar{\alpha}, \bar{\alpha} \rangle} = \frac{2\langle \pm \tilde{b}_k \pm \tilde{b}_m, \pm \tilde{b}_i \pm \tilde{b}_j \rangle}{\langle \pm \tilde{b}_i \pm \tilde{b}_j, \pm \tilde{b}_i \pm \tilde{b}_j \rangle}
\]
must always be an integer. As a result, the set $\Delta = \{ \pm \tilde{e}_i \pm \tilde{e}_j, i \neq j \} \cup \{ \pm \tilde{e}_i \}$ is an abstract root system.

We call this type of abstract root system $B_n$. Suppose $n = 2$. In this case, $V = \mathbb{R}^2$ and
\[
\Delta = \{ \tilde{e}_1 + \tilde{e}_2, \tilde{e}_1 - \tilde{e}_2, -\tilde{e}_1 + \tilde{e}_2, -\tilde{e}_1 - \tilde{e}_2, \tilde{e}_1, -\tilde{e}_1, \tilde{e}_2, -\tilde{e}_2 \}.
\]
We will once again choose the standard basis for $\mathbb{R}^2$ where $\tilde{e}_1 = (1, 0)$ and $\tilde{e}_2 = (0, 1)$. Hence,
\[
\Delta = \{ (1, 1), (1, -1), (-1, 1), (-1, -1), (1, 0), (-1, 0), (0, 1), (0, -1) \},
\]
which is depicted graphically in figure 4.
Figure 4. $B_2$ Abstract Root System

We will now consider the sizes and further classifications of abstract root systems. We begin with a simple definition.

**Definition 5.** An abstract root system is **reducible** if it can be represented as a disjoint union of two abstract root systems $\Delta = \Delta' \cup \Delta''$, and each element of $\Delta'$ is orthogonal to each element of $\Delta''$. We say that $\Delta$ is **irreducible** if it admits no such decomposition.

This definition motivates us to test our familiar root systems $A_2$ and $B_2$ to determine whether they are reducible. It is clear from figure 2 that no root system $\Delta_{A_2}$ of type $A_2$ is reducible since no two vectors in any $\Delta_{A_2}$ are orthogonal, and so there cannot possibly be two smaller root systems $\Delta'_{A_2}, \Delta''_{A_2} \subset \Delta_{A_2}$ where each element in $\Delta'_{A_2}$ orthogonal to each element in $\Delta''_{A_2}$.

Now consider root systems of type $B_2$. From figure 4, it is equally obvious that there exists no reducible root system $\Delta_{B_2}$ of type $B_2$. In this case, each of the eight vectors in $\Delta_{B_2}$ is orthogonal to only one of the other vectors in $\Delta_{B_2}$. Hence, we cannot find two sets $\Delta'_{B_2}$ and $\Delta''_{B_2}$ that are orthogonal to each other.

An example of a reducible root system in $\mathbb{R}^2$ is $A_1 \oplus A_1$, which is the union of two $A_1$ root systems. Suppose $\Delta_{A_1 \oplus A_1}$ is an abstract root system of type $A_1 \oplus A_1$. That is, $\Delta_{A_1 \oplus A_1} = \{\vec{e}_1, -\vec{e}_1, \vec{e}_2, -\vec{e}_2\}$. Let $\Delta'_{A_1 \oplus A_1} = \{\vec{e}_1, -\vec{e}_1\}$ and $\Delta''_{A_1 \oplus A_1} = \{\vec{e}_2, -\vec{e}_2\}$. It is obvious that these two sets are orthogonal.

To further classify abstract root systems, we will prove some elementary theorems about them.

(1) If $\alpha \in \Delta$, then $-\alpha \in \Delta$.

(2) If $\alpha \in \Delta$ and $\pm \frac{1}{7} \alpha$ is not in $\Delta$, then the only possible elements of $\Delta \cup \{0\}$ proportional to $\alpha$ are $\pm \alpha$, $\pm 2\alpha$, and $0$.

(3) If $\alpha$ is in $\Delta$ and $\beta \in \Delta \cup 0$, then

\[ n(\alpha, \beta) := \frac{2(\beta, \alpha)}{\langle \alpha, \alpha \rangle} = 0, \pm 1, \pm 2, \pm 3 \text{ or } \pm 4, \]

and $\pm 4$ can only occur if $\beta = \pm 2\alpha$.

Proof. (1) Consider

\[ s_{\alpha}(\alpha) = \alpha - \frac{2(\alpha, \alpha)}{\langle \alpha, \alpha \rangle} \alpha = -\alpha. \]

By the definition of an abstract root system, the reflections map the set $\Delta$ to itself. Hence, if $\alpha \in \Delta$, then $-\alpha \in \Delta$.

(2) To prove the second property, we will use the fact that $n(\alpha, \beta)$ must be an integer. Hence, if $k \in \mathbb{R}$,

\[ \frac{2(\alpha k \alpha, \alpha \alpha)}{\langle \alpha, \alpha \rangle} \in \mathbb{Z} \quad \text{and} \quad \frac{2(\alpha \alpha k \alpha \alpha)}{\langle \alpha \alpha, \alpha \alpha \rangle} \in \mathbb{Z}. \]

By the properties of the inner product, it follows that $2/k$ and $2k$ are both integers. We know that either $k = 0$ or $|k| \geq 1/2$ for $2k \in \mathbb{Z}$. We also know that $2/k$ can only be an integer larger than 4 if $|k| < 1/2$. Hence, it suffices to find the $k$ that satisfy the equation $2/k = c$, where $c = \{\pm 1, \pm 2, \pm 3, \pm 4\}$. We can rewrite this equation as $k = 2/c$ to see that $k = \{\pm 2, \pm 1, \pm 2/3, \pm 1/2\}$. Reject $k = \pm 2/3$, since $4/3 \notin \mathbb{Z}$, and $k = \pm 1/2$ by the statement of the theorem. Consequently, the only possible elements of $\Delta \cup \{0\}$ proportional to $\alpha$
are $\pm \alpha$, $\pm 2\alpha$, and $\bar{0}$.

(3) The third property is proved using the Cauchy-Schwarz inequality, which states that

$$\left| \langle \widetilde{\beta}, \widetilde{\alpha} \rangle \right| \leq \| \widetilde{\alpha} \| \cdot \| \widetilde{\beta} \| .$$

We can rewrite (17) as

$$\left| \langle \widetilde{\beta}, \widetilde{\alpha} \rangle \right| \leq \langle \widetilde{\alpha}, \widetilde{\alpha} \rangle^{1/2} \cdot \langle \widetilde{\beta}, \widetilde{\beta} \rangle^{1/2} .$$

Squaring both sides we notice that

$$\langle \widetilde{\beta}, \widetilde{\alpha} \rangle^2 \leq \langle \widetilde{\alpha}, \widetilde{\alpha} \rangle \cdot \langle \widetilde{\beta}, \widetilde{\beta} \rangle .$$

By the bilinearity of the inner product,

$$\langle \widetilde{\beta}, \widetilde{\alpha} \rangle \cdot \langle \widetilde{\alpha}, \widetilde{\beta} \rangle \leq \langle \widetilde{\alpha}, \widetilde{\alpha} \rangle \cdot \langle \widetilde{\beta}, \widetilde{\beta} \rangle .$$

It therefore follows that

$$\frac{2\langle \widetilde{\beta}, \alpha \rangle}{\langle \widetilde{\alpha}, \alpha \rangle} \cdot \frac{2\langle \widetilde{\alpha}, \beta \rangle}{\langle \widetilde{\beta}, \beta \rangle} \leq 4 .$$

Once again, the fractions

$$\frac{2\langle \widetilde{\beta}, \alpha \rangle}{\langle \widetilde{\alpha}, \alpha \rangle} \quad \text{and} \quad \frac{2\langle \widetilde{\alpha}, \beta \rangle}{\langle \widetilde{\beta}, \beta \rangle}$$

must be integers. When taken together, their product must be less than or equal to four, so each of these fractions can only be 0, ±1, ±2, ±3, or ±4.

Suppose that

$$\frac{2\langle \widetilde{\beta}, \alpha \rangle}{\langle \widetilde{\alpha}, \alpha \rangle} = \pm 4 .$$

In this case, equality holds in the Cauchy-Schwarz inequality, so $\widetilde{\alpha}$ and $\widetilde{\beta}$ are proportional. In addition, by (17),

$$\frac{2\langle \widetilde{\alpha}, \beta \rangle}{\langle \widetilde{\beta}, \beta \rangle} = \pm 1 .$$

As a result, $2\langle \widetilde{\beta}, \alpha \rangle = 4\| \alpha \|^2$ and $2\langle \widetilde{\alpha}, \beta \rangle = 2\| \beta \|^2$, so $\| \beta \| = 2\| \alpha \|$. Since $\alpha$ is proportional to $\beta$, it is clear that $\beta = \pm 2\alpha$.

$\square$

Property (3) of theorem 6 limits the magnitude of $n(\alpha, \beta)$. This leads us to consider the possible values of $n(\alpha, \beta)$ for the familiar, two dimensional root systems of type $A_2$, $B_2$, and $A_1 \oplus A_1$. Brute force calculations show that in each of these three cases, $n(\alpha, \beta)$ can only be 0, ±1, or ±2. We must now try to find abstract root systems in $V = \mathbb{R}^2$ which allow $n(\alpha, \beta)$
to equal ±3 or ±4. To accomplish this goal, we must first prove another theorem.

**Theorem 7.** Let $\Delta$ be an abstract root system in $V$.

1. If $\bar{\alpha}$ and $\bar{\beta}$ are in $\Delta$, and $\langle \bar{\alpha}, \bar{\beta} \rangle > 0$, then $\bar{\alpha} - \bar{\beta}$ is a root or 0. If $\langle \bar{\alpha}, \bar{\beta} \rangle < 0$, then $\bar{\alpha} + \bar{\beta}$ is a root or 0.

2. Let $\bar{\alpha} \in \Delta$ and $\bar{\beta} \in \Delta \cup \{\bar{0}\}$. If $\bar{\beta} + n\bar{\alpha}, \bar{\beta} + (n+1)\bar{\alpha} \in \Delta \cup \{\bar{0}\}$, then $\bar{\beta} + (n + 1)\bar{\alpha}$ must also be in $\Delta \cup \{\bar{0}\}$.

**Proof.**

(1) Consider the following two reflections:

\begin{align}
   s_{\bar{\alpha}}(\bar{\beta}) &= \bar{\beta} - n(\bar{\alpha}, \bar{\beta})\bar{\alpha} \\
   s_{\bar{\beta}}(\bar{\alpha}) &= \bar{\alpha} - n(\bar{\beta}, \bar{\alpha})\bar{\beta}
\end{align}

For $\bar{\alpha} - \bar{\beta}$ to be a root, it suffices that $n(\bar{\beta}, \bar{\alpha}) = 1$ or $n(\bar{\alpha}, \bar{\beta}) = 1$. For $\bar{\alpha} + \bar{\beta}$ to be a root, it also suffices that $n(\bar{\beta}, \bar{\alpha}) = -1$ or $n(\bar{\alpha}, \bar{\beta}) = -1$. By equation (18), $\left| n(\bar{\alpha}, \bar{\beta}) \cdot n(\bar{\beta}, \bar{\alpha}) \right| \leq 4$. Hence, the following are the possible values for $n(\bar{\alpha}, \bar{\beta})$ and $n(\bar{\beta}, \bar{\alpha})$:

<table>
<thead>
<tr>
<th>$n(\bar{\alpha}, \bar{\beta})$</th>
<th>$n(\bar{\beta}, \bar{\alpha})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>±1</td>
<td>±1, ±2, ±3, ±4</td>
</tr>
<tr>
<td>±2</td>
<td>±1, ±2</td>
</tr>
<tr>
<td>±3</td>
<td>±1</td>
</tr>
<tr>
<td>±4</td>
<td>±1</td>
</tr>
</tbody>
</table>

When $n(\bar{\alpha}, \bar{\beta}) = ±2$ and $n(\bar{\beta}, \bar{\alpha}) = ±2$, $\left| n(\bar{\alpha}, \bar{\beta}) \right| = \left| n(\bar{\beta}, \bar{\alpha}) \right|$. Hence $\langle \bar{\alpha}, \bar{\alpha} \rangle = \langle \bar{\beta}, \bar{\beta} \rangle$. We know by the previous theorem that $\bar{\alpha}$ and $\bar{\beta}$ are proportional. Thus, $\bar{\beta}$ must be ±$\bar{\alpha}$. In every other case, either $n(\bar{\alpha}, \bar{\beta})$ or $n(\bar{\beta}, \bar{\alpha})$ must be ±1. If $\langle \bar{\alpha}, \bar{\beta} \rangle > 0$, then $n(\bar{\alpha}, \bar{\beta}) > 0$ and $n(\bar{\beta}, \bar{\alpha}) > 0$. Therefore, the reflections in equations (19) and (20) yield either $\bar{\beta} - \bar{\alpha}$ or $\bar{\alpha} - \bar{\beta}$. If $\langle \bar{\alpha}, \bar{\beta} \rangle < 0$, then $n(\bar{\alpha}, \bar{\beta}) < 0$ and $n(\bar{\beta}, \bar{\alpha}) < 0$. Now, these reflections yield $\bar{\alpha} + \bar{\beta}$.

(2) We will prove the second statement by contradiction. Suppose that $\bar{\beta} + n\bar{\alpha}, \bar{\beta} + (n+2)\bar{\alpha} \in \Delta \cup \{\bar{0}\}$, but $\bar{\beta} + (n+1)\bar{\alpha} \not\in \Delta \cup \{\bar{0}\}$. Hence, we assume that there is a gap in the set of elements of $\Delta \cup \{\bar{0}\}$ of the form $\bar{\beta} + n\bar{a}$. We know that $\bar{\alpha} \in \Delta$, so by the first part of theorem 7, either

\begin{align}
   \bar{\beta} + (n + 2)\bar{\alpha} - \bar{\alpha} &= \bar{\beta} + (n + 1)\bar{\alpha} \in \Delta \cup \{\bar{0}\}
\end{align}
if \( \langle \vec{\beta} + (n + 2)\vec{\alpha}, \vec{\alpha} \rangle > 0 \), or

\[
\vec{\beta} + n\vec{\alpha} + \vec{\alpha} = \vec{\beta} + (n + 1)\vec{\alpha} \in \Delta \cup \{ \vec{0} \}
\]

if \( \langle \vec{\beta} + n\vec{\alpha}, \vec{\alpha} \rangle < 0 \). By simplifying these conditions, we observe that \( \vec{\beta} + (n + 1)\vec{\alpha} \in \Delta \cup \{ \vec{0} \} \) if \( \langle \vec{\beta}, \vec{\alpha} \rangle > -(n + 2)\langle \vec{\alpha}, \vec{\alpha} \rangle \) or \( \langle \vec{\beta}, \vec{\alpha} \rangle < -n\langle \vec{\alpha}, \vec{\alpha} \rangle \). These two conditions cover all possibilities for \( \langle \vec{\alpha}, \vec{\beta} \rangle \), so \( \vec{\beta} + (n + 1)\vec{\alpha} \in \Delta \cup \{ \vec{0} \} \), which contradicts our original proposition.

\[ \square \]

We will conclude by taking advantage of the Euclidean geometry to describe the geometry of abstract root systems. Recall that for the standard inner product in \( \mathbb{R}^n \), the number \( \langle \vec{\alpha}, \vec{\alpha} \rangle = \| \vec{\alpha} \|^2 \) is the square of the length of the vector. Hence, \( n(\vec{\alpha}, \vec{\beta}) \) can be written as

\[
n(\vec{\alpha}, \vec{\beta}) = 2\frac{\| \vec{\beta} \|}{\| \vec{\alpha} \|} \cos \phi,
\]

where \( \phi \) is the angle between \( \vec{\alpha} \) and \( \vec{\beta} \). Then we have

\[
\left| n(\vec{\alpha}, \vec{\beta}) \cdot n(\vec{\beta}, \vec{\alpha}) \right| = 4 \cos^2 \phi.
\]

By applying theorem 6, we can find all of the possible values for \( \phi \), as shown in the table below.

| \( n(\vec{\alpha}, \vec{\beta}) \) | \( n(\vec{\beta}, \vec{\alpha}) \) | \( |n(\vec{\alpha}, \vec{\beta}) \cdot n(\vec{\beta}, \vec{\alpha})| \) | \( \cos \phi \) | \( \phi \) |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 90° |
| ±1 | ±1, ±2, ±3, ±4 | 1, 2, 3, 4 | 1/2, 1/\sqrt{2}, \sqrt{3}/2, 1 | 60°, 45°, 30°, 0° |
| ±2 | ±1, ±2 | 2, 4 | 1/\sqrt{2}, 1 | 45°, 0° |
| ±3 | ±1 | 3 | \sqrt{3}/2 | 30° |
| ±4 | ±1 | 4 | 1 | 0° |

Consequently, the angle \( \phi \) between two nonproportional elements of an abstract root system can only be 30°, 45°, 60°, or 90°.

The relative lengths of any two vectors can also be predicted. Equation (23) also implies that

\[
n(\vec{\beta}, \vec{\alpha}) = 2\frac{\| \vec{\alpha} \|}{\| \vec{\beta} \|} \cos \phi,
\]

so we will now calculate all possible ratios \( \| \vec{\beta} \| / \| \vec{\alpha} \| \) and \( \| \vec{\alpha} \| / \| \vec{\beta} \| \) of the roots for a fixed angle \( \phi \). The possible relative lengths are therefore those values that satisfy both ratios, as shown in the following table.
\[
\begin{array}{|c|c|c|}
\hline
\phi & \|\vec{\alpha}\| / \|\vec{\beta}\| & \text{relative length } \geq 1 \\
\hline
30^\circ & \frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{3}{\sqrt{3}} & \frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{3}{\sqrt{3}} , \frac{1}{\sqrt{2}} \frac{3}{\sqrt{2}} \frac{1}{\sqrt{2}} \frac{1}{\sqrt{3}} \\
45^\circ & \frac{1}{\sqrt{2}}, \frac{\sqrt{2}}{\sqrt{2}} \frac{2}{\sqrt{2}} \frac{\sqrt{2}}{\sqrt{2}} \frac{2}{\sqrt{2}} \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \\
60^\circ & 1, 2, 3, 4 & 1, 1/2, 1/3, 1/4 \\
90^\circ & 1/2, 1, 3/2, 2 & 1, 2, 1/3, 2/3, 1/2 \\
\hline
\end{array}
\]

We now have all the tools we need to describe all possible root systems in \( V = \mathbb{R}^2 \). We have already encountered three of them—\( A_1 \oplus A_1, A_2 \), and \( B_2 \). Recall that for \( A_1 \oplus A_1 \), 4 roots meet at \( 90^\circ \) angles. For \( A_2 \) abstract root systems, 6 roots meet with \( 60^\circ \) angles between adjacent roots, so the only relative length they can have is 1. For \( B_2 \) systems, 8 roots meet with \( 45^\circ \) angles between adjacent ones, so the possible relative lengths are \( 1/\sqrt{2} \) for those vectors with \( 45^\circ \) between them and 1 for those vectors with \( 90^\circ \) between them. If the relative lengths of vectors that intersect at \( 45^\circ \) angles in \( B_2 \) is instead \( \sqrt{2} \), we have a fifth abstract root system, \( C_2 \). In effect, this is just a rotated version \( B_2 \). If we superimpose the \( B_2 \) and \( C_2 \) root systems, we get a fifth abstract root system in \( \mathbb{R}^2 \) known as \( BC_2 \). Finally, if we take the angle between 12 adjacent roots to be \( 30^\circ \) apart, we see that we get relative length to be \( \sqrt{3} \) between adjacent roots and 1 between alternating roots. We call this sixth root system \( G_2 \). The root systems \( C_2, BC_2, \) and \( G_2 \) are depicted below.

**Figure 6.** \( C_2 \) Abstract Root System
Figure 7. $BC_2$ Abstract Root System

Figure 8. $G_2$ Abstract Root System
No other root systems can possibly exist in $\mathbb{R}^2$. Thus, maximum number of roots in any root system on $\mathbb{R}^2$ is 12.

References