REFLECTIONS IN A EUCLIDEAN SPACE

YOUR NAME HERE

18.099 - 18.06 CI.
Due on Monday, May 10 in class.

Write a paper proving the statements formulated below. Add your own examples, asides and discussions whenever needed.

Let \( V \) be a finite dimensional real linear space.

**Definition 1.** A function \( \langle \cdot, \cdot \rangle : V \times V \to \mathbb{R} \) is a bilinear form on \( V \) if for all \( x_1, x_2, x, y_1, y_2, y \in V \) and all \( k \in \mathbb{R} \),
\[
\langle x_1 + kx_2, y \rangle = \langle x_1, y \rangle + k\langle x_2, y \rangle, \quad \text{and}
\]
\[
\langle x, y_1 + ky_2 \rangle = \langle x, y_1 \rangle + k\langle x, y_2 \rangle.
\]

**Definition 2.** A bilinear form \( \langle \cdot, \cdot \rangle \) in \( V \) is symmetric if \( \langle x, y \rangle = \langle y, x \rangle \) for all \( x, y \in V \). A symmetric bilinear form is nondegenerate if \( \langle a, x \rangle = 0 \) for all \( x \in V \) implies \( a = 0 \). It is positive definite if \( \langle x, x \rangle > 0 \) for any nonzero \( x \in V \). An inner product on \( V \) is a symmetric positive definite bilinear form on \( V \).

**Theorem 3.** Define a bilinear form on \( V = \mathbb{R}^n \) by \( \langle e_i, e_j \rangle = \delta_{ij} \), where \( \{e_i\}_{i=1}^n \) is a basis in \( V \). Then \( \langle \cdot, \cdot \rangle \) is an inner product in \( V \).

**Definition 4.** An Euclidean space is a finite dimensional real linear space with an inner product.

**Theorem 5.** Any \( n \)-dimensional Euclidean space \( V \) has a basis \( \{e_i\}_{i=1}^n \) such that \( \langle e_i, e_j \rangle = \delta_{ij} \).

Hint: Use the Gram-Schmidt orthogonalization process.

Below \( V = \mathbb{R}^n \) is a Euclidean space with the inner product \( \langle \cdot, \cdot \rangle \).

**Definition 6.** Two vectors \( x, y \in V \) are orthogonal if \( \langle x, y \rangle = 0 \). Two subspaces \( U, W \in V \) are orthogonal if \( \langle x, y \rangle = 0 \) for all \( x \in U \) and \( y \in W \).

Check that if \( U \) and \( W \) are orthogonal subspaces in \( V \), then \( \dim(U) + \dim(W) = \dim(U + W) \).

**Definition 7.** The orthogonal complement of the subspace \( U \subset V \) is the subspace \( U^\perp = \{ y \in V : \langle x, y \rangle = 0, \text{ for all } x \in U \} \).

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**Definition 8.** A hyperplane $H_x \subset V$ is the orthogonal complement to the one-dimensional subspace in $V$ spanned by $x \in V$.

**Theorem 9.** (Cauchy-Schwartz). For any $x, y \in V$, 

$$\langle x, y \rangle^2 \leq \langle x, x \rangle \cdot \langle y, y \rangle,$$

and equality holds if and only if the vectors $x$ and $y$ are linearly dependent.

We will be interested in the linear mappings that respect inner products.

**Definition 10.** An orthogonal operator in $V$ is a linear automorphism $f : V \rightarrow V$ such that $\langle f(x), f(y) \rangle = \langle x, y \rangle$ for all $x, y \in V$.

**Theorem 11.** If $f_1, f_2$ are orthogonal operators in $V$, then so are the inverses $f_1^{-1}$ and $f_2^{-1}$ and the composition $f_1 \circ f_2$. The identity mapping is orthogonal.

**Remark 12.** The above theorem says that orthogonal operators in a Euclidean space form a group, that is, a set closed with respect to compositions, containing an inverse to each element, and containing an identity operator.

**Example 13.** Describe the set of $2 \times 2$ matrices of all orthogonal operators in $\mathbb{R}^2$, and check that they form a group with respect to the matrix multiplication.

Now we are ready to introduce the notion of a reflection in a Euclidean space. A reflection in $V$ is a linear mapping $s : V \rightarrow V$ which sends some nonzero vector $\alpha \in V$ to its negative and fixes pointwise the hyperplane $H_\alpha$ orthogonal to $\alpha$. To indicate this vector, we will write $s = s_\alpha$. The use of Greek letters for vectors is traditional in this context.

**Definition 14.** A reflection in $V$ with respect to a vector $\alpha \in V$ is defined by the formula:

$$s_\alpha(x) = x - \frac{2 \langle x, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha.$$

**Theorem 15.** With the above definition, we have:

1. $s_\alpha(\alpha) = -\alpha$ and $s_\alpha(x) = x$ for any $x \in H_\alpha$;
2. $s_\alpha$ is an orthogonal operator;
3. $s_\alpha^2 = Id$.

Therefore, reflections generate a group: their compositions are orthogonal operators by Theorem 11, and an inverse of a reflection is equal to itself by Theorem 15. Below we consider some basic examples of subgroups of orthogonal operators obtained by repeated application of reflections.

**Example 16.** Consider the group $S_n$ of permutations of $n$ numbers. It is generated by transpositions $t_{ij}$ where $i \neq j$ are two numbers between 1 and $n$, and $t_{ij}$ sends $i$ to $j$ and $j$ to $i$, while preserving all other numbers.
The compositions of all such transpositions form $S_n$. Define a set of linear mappings $T_{ij} : \mathbb{R}^n \to \mathbb{R}^n$ in an orthonormal basis $\{e_i\}_{i=1}^n$ by

$$T_{ij}e_i = e_j; \quad T_{ij}e_j = e_i; \quad T_{ij}e_k = e_k, k \neq i, j.$$ 

Then, since any element $\sigma \in S_n$ is a composition of transpositions, it defines a linear automorphism of $\mathbb{R}^n$ equal to the composition of the linear mappings defined above.

1. Check that $T_{ij}$ acts as a reflection with respect to the vector $e_i - e_j \in \mathbb{R}^n$.
2. Check that any element $\sigma$ of $S_n$ fixes pointwise the line in $\mathbb{R}^n$ spanned by $e_1 + e_2 + \ldots e_n$.
3. Let $n = 3$. Describe the action of each element (how many are there?) of $S_3$ in $\mathbb{R}^3$ and in the plane $U$ orthogonal to $e_1 + e_2 + e_3$. Example 13 lists all matrices of orthogonal operators in $\mathbb{R}^2$. Identify among them the matrices corresponding to the elements of $S_3$ acting in $U$. Check that the product of two reflections is a rotation.

Example 17. The action of $S_n$ in $\mathbb{R}^n$ described above can be composed with the reflections $\{P_i\}_{i=1}^n$, sending $e_i$ to its negative and fixing all other elements of the basis $e_k, k \neq i$.

1. Check that the obtained set of orthogonal operators has no nonzero fixed points (elements $x \in \mathbb{R}^n$ such that $f(x) = x$ for all $f$ in the set).
2. How many distinct orthogonal operators can be constructed in this way for $n = 2$ and $n = 3$?
3. In case $n = 2$, identify the matrices of the obtained orthogonal operators among those listed in Example 13.

Remark 18. The two examples above correspond to the series $A_{n-1}$ and $B_n$ in the classification of finite reflection groups.