**An overview of key ideas**

This is an overview of linear algebra given at the start of a course on the mathematics of engineering.

Linear algebra progresses from vectors to matrices to subspaces.

**Vectors**

What do you do with vectors? Take combinations.

We can multiply vectors by scalars, add, and subtract. Given vectors \(\mathbf{u}, \mathbf{v}\) and \(\mathbf{w}\) we can form the linear combination

\[x_1 \mathbf{u} + x_2 \mathbf{v} + x_3 \mathbf{w} = \mathbf{b}.
\]

An example in \(\mathbb{R}^3\) would be:

\[
\begin{bmatrix}
1 & 0 & 0 \\
-1 & 0 & 1 \\
0 & 1 & -1
\end{bmatrix}
\]

\[
\begin{bmatrix}
\mathbf{u} \\
\mathbf{v} \\
\mathbf{w}
\end{bmatrix}
\]

The collection of all multiples of \(\mathbf{u}\) forms a line through the origin. The collection of all multiples of \(\mathbf{v}\) forms another line. The collection of all combinations of \(\mathbf{u}\) and \(\mathbf{v}\) forms a plane. Taking all combinations of some vectors creates a subspace.

We could continue like this, or we can use a matrix to add in all multiples of \(\mathbf{w}\).

**Matrices**

Create a matrix \(A\) with vectors \(\mathbf{u}, \mathbf{v}\) and \(\mathbf{w}\) in its columns:

\[
A = \begin{bmatrix}
1 & 0 & 0 \\
-1 & 1 & 0 \\
0 & -1 & 1
\end{bmatrix}.
\]

The product:

\[
A\mathbf{x} = \begin{bmatrix}
1 & 0 & 0 \\
-1 & 1 & 0 \\
0 & -1 & 1
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix}
x_1 \\
-x_1 + x_2 \\
-x_2 + x_3
\end{bmatrix}
\]

equals the sum \(x_1 \mathbf{u} + x_2 \mathbf{v} + x_3 \mathbf{w} = \mathbf{b}\). The product of a matrix and a vector is a combination of the columns of the matrix. (This particular matrix \(A\) is a difference matrix because the components of \(A\mathbf{x}\) are differences of the components of that vector.)

When we say \(x_1 \mathbf{u} + x_2 \mathbf{v} + x_3 \mathbf{w} = \mathbf{b}\) we’re thinking about multiplying numbers by vectors; when we say \(A\mathbf{x} = \mathbf{b}\) we’re thinking about multiplying a matrix (whose columns are \(\mathbf{u}, \mathbf{v}\) and \(\mathbf{w}\)) by the numbers. The calculations are the same, but our perspective has changed.
For any input vector \( x \), the output of the operation “multiplication by \( A \)” is some vector \( b \):

\[
A \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}.
\]

A deeper question is to start with a vector \( b \) and ask “for what vectors \( x \) does \( Ax = b \)?” In our example, this means solving three equations in three unknowns. Solving:

\[
Ax = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}
\]

is equivalent to solving:

\[
\begin{aligned}
x_1 &= b_1 \\
x_2 - x_1 &= b_2 \\
x_3 - x_2 &= b_3.
\end{aligned}
\]

We see that \( x_1 = b_1 \) and so \( x_2 \) must equal \( b_1 + b_2 \). In vector form, the solution is:

\[
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_1 + b_2 \\ b_1 + b_2 + b_3 \end{bmatrix}.
\]

But this just says:

\[
x = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix},
\]

or \( x = A^{-1}b \). If the matrix \( A \) is invertible, we can multiply on both sides by \( A^{-1} \) to find the unique solution \( x \) to \( Ax = b \). We might say that \( A \) represents a transform \( x \rightarrow b \) that has an inverse transform \( b \rightarrow x \).

In particular, if \( b = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \) then \( x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \).

The second example has the same columns \( u \) and \( v \) and replaces column vector \( w \):

\[
C = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.
\]

Then:

\[
Cx = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix}
\]

and our system of three equations in three unknowns becomes circular.
Where before $Ax = 0$ implied $x = 0$, there are non-zero vectors $x$ for which $Cx = 0$. For any vector $x$ with $x_1 = x_2 = x_3$, $Cx = 0$. This is a significant difference; we can't multiply both sides of $Cx = 0$ by an inverse to find a non-zero solution $x$.

The system of equations encoded in $Cx = b$ is:

\[
\begin{align*}
x_1 - x_3 &= b_1 \\
x_2 - x_1 &= b_2 \\
x_3 - x_2 &= b_3.
\end{align*}
\]

If we add these three equations together, we get:

\[0 = b_1 + b_2 + b_3.\]

This tells us that $Cx = b$ has a solution $x$ only when the components of $b$ sum to 0. In a physical system, this might tell us that the system is stable as long as the forces on it are balanced.

**Subspaces**

Geometrically, the columns of $C$ lie in the same plane (they are dependent; the columns of $A$ are independent). There are many vectors in $\mathbb{R}^3$ which do not lie in that plane. Those vectors cannot be written as a linear combination of the columns of $C$ and so correspond to values of $b$ for which $Cx = b$ has no solution $x$. The linear combinations of the columns of $C$ form a two dimensional subspace of $\mathbb{R}^3$.

This plane of combinations of $u$, $v$ and $w$ can be described as “all vectors $Cx$”. But we know that the vectors $b$ for which $Cx = b$ satisfy the condition $b_1 + b_2 + b_3 = 0$. So the plane of all combinations of $u$ and $v$ consists of all vectors whose components sum to 0.

If we take all combinations of:

\[
\begin{bmatrix}
1 \\
-1 \\
0
\end{bmatrix}, \begin{bmatrix}
0 \\
1 \\
-1
\end{bmatrix}, \text{ and } \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}
\]

we get the entire space $\mathbb{R}^3$; the equation $Ax = b$ has a solution for every $b$ in $\mathbb{R}^3$. We say that $u$, $v$ and $w$ form a basis for $\mathbb{R}^3$.

A basis for $\mathbb{R}^n$ is a collection of $n$ independent vectors in $\mathbb{R}^n$. Equivalently, a basis is a collection of $n$ vectors whose combinations cover the whole space. Or, a collection of vectors forms a basis whenever a matrix which has those vectors as its columns is invertible.

A vector space is a collection of vectors that is closed under linear combinations. A subspace is a vector space inside another vector space; a plane through the origin in $\mathbb{R}^3$ is an example of a subspace. A subspace could be equal to the space it’s contained in; the smallest subspace contains only the zero vector.

The subspaces of $\mathbb{R}^3$ are:
• the origin,
• a line through the origin,
• a plane through the origin,
• all of $\mathbb{R}^3$.

Conclusion

When you look at a matrix, try to see “what is it doing?”

Matrices can be rectangular; we can have seven equations in three unknowns. Rectangular matrices are not invertible, but the symmetric, square matrix $A^TA$ that often appears when studying rectangular matrices may be invertible.