Exam 1 review

This lecture is a review for the exam. The majority of the exam is on what we’ve learned about rectangular matrices.

Sample question 1

Suppose $u, v$ and $w$ are non-zero vectors in $\mathbb{R}^7$. They span a subspace of $\mathbb{R}^7$. What are the possible dimensions of that vector space?

The answer is 1, 2 or 3. The dimension can’t be higher because a basis for this subspace has at most three vectors. It can’t be 0 because the vectors are non-zero.

Sample question 2

Suppose a 5 by 3 matrix $R$ in reduced row echelon form has $r = 3$ pivots.

1. What’s the nullspace of $R$?
   
   Since the rank is 3 and there are 3 columns, there is no combination of the columns that equals 0 except the trivial one. $N(R) = \{0\}$.

2. Let $B$ be the 10 by 3 matrix $\begin{bmatrix} R \\ 2R \end{bmatrix}$. What’s the reduced row echelon form of $B$?
   
   Answer: $\begin{bmatrix} R \\ 0 \end{bmatrix}$.

3. What is the rank of $B$?
   
   Answer: 3.

4. What is the reduced row echelon form of $C = \begin{bmatrix} R & R \\ R & 0 \end{bmatrix}$?
   
   When we perform row reduction we get:
   
   $\begin{bmatrix} R & R \\ R & 0 \end{bmatrix} \rightarrow \begin{bmatrix} R & R \\ 0 & -R \end{bmatrix} \rightarrow \begin{bmatrix} R & 0 \\ 0 & -R \end{bmatrix} \rightarrow \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}$.

   Then we might need to move some zero rows to the bottom of the matrix.

5. What is the rank of $C$?
   
   Answer: 6.

6. What is the dimension of the nullspace of $C^T$?
   
   $m = 10$ and $r = 6$ so $\dim N(C^T) = 10 - 6 = 4$. 

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Sample question 3

Suppose we know that \( Ax = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} \) and that:

\[
\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\]

is a complete solution.

Note that in this problem we don’t know what \( A \) is.

1. What is the shape of the matrix \( A \)?
   
   Answer: 3 by 3, because \( \mathbf{x} \) and \( \mathbf{b} \) both have three components.

2. What’s the dimension of the row space of \( A \)?
   
   From the complete solution we can see that the dimension of the nullspace of \( A \) is 2, so the rank of \( A \) must be \( 3 - 2 = 1 \).

3. What is \( A \)?
   
   Because the second and third components of the particular solution \( \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \)
   
   are zero, we see that the first column vector of \( A \) must be \( \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \).

   Knowing that \( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \) is in the nullspace tells us that the third column of \( A \)
   
   must be 0. The fact that \( \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \) is in the nullspace tells us that the second column must be the negative of the first. So,

   \[
   A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ 1 & -1 & 0 \end{bmatrix}.
   \]

   If we had time, we could check that this \( A \) times \( \mathbf{x} \) equals \( \mathbf{b} \).

4. For what vectors \( \mathbf{b} \) does \( Ax = b \) have a solution \( \mathbf{x} \)?
   
   This equation has a solution exactly when \( \mathbf{b} \) is in the column space of \( A \),

   so when \( \mathbf{b} \) is a multiple of \( \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \). This makes sense; we know that the rank of \( A \) is 1 and the nullspace is large.

   In contrast, we might have had \( r = m \) or \( r = n \).
Sample question 4

Suppose:

\[
B = CD = \begin{bmatrix}
1 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 & -1 & 2 \\
0 & 1 & 1 & -1 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

Try to answer the questions below without performing this matrix multiplication \(CD\).

1. Give a basis for the nullspace of \(B\).

The matrix \(B\) is 3 by 4, so \(N(B) \subseteq \mathbb{R}^4\). Because \(C = \begin{bmatrix}
1 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0
\end{bmatrix}\) is invertible, the nullspace of \(B\) is the same as the nullspace of \(D = \begin{bmatrix}
1 & 0 & -1 & 2 \\
0 & 1 & 1 & -1 \\
0 & 0 & 0 & 0
\end{bmatrix}\). Matrix \(D\) is in reduced form, so its special solutions form a basis for \(N(D) = N(B)\):

\[
\begin{bmatrix}
1 \\
-1 \\
1 \\
0
\end{bmatrix}, \begin{bmatrix}
-2 \\
1 \\
0 \\
1
\end{bmatrix}.
\]

These vectors are independent, and if time permits we can multiply to check that they are in \(N(B)\).

2. Find the complete solution to \(Bx = 0\).

We can now describe any vector in the nullspace, so all we need to do is find a particular solution. There are many possible particular solutions; the simplest one is given below.

One way to solve this is to notice that \(C \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\) and then find a vector \(x\) for which \(Dx = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\). Another approach is to notice that the first column of \(B = CD\) is \(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\). In either case, we get the complete solution:

\[
x = \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix} + c \begin{bmatrix}
-1 \\
1 \\
0
\end{bmatrix} + d \begin{bmatrix}
-2 \\
1 \\
0
\end{bmatrix}.
\]

Again, we can check our work by multiplying.
Short questions

There may not be true/false questions on the exam, but it’s a good idea to review these:

1. Given a square matrix $A$ whose nullspace is just $\{0\}$, what is the nullspace of $A^T$?
   
   $N(A^T)$ is also $\{0\}$ because $A$ is square.

2. Do the invertible matrices form a subspace of the vector space of 5 by 5 matrices?
   
   No. The sum of two invertible matrices may not be invertible. Also, 0 is not invertible, so is not in the collection of invertible matrices.

3. True or false: If $B^2 = 0$, then it must be true that $B = 0$.
   
   False. We could have $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

4. True or false: A system $Ax = b$ of $n$ equations with $n$ unknowns is solvable for every right hand side $b$ if the columns of $A$ are independent.
   
   True. $A$ is invertible, and $x = A^{-1}b$ is a (unique) solution.

5. True or false: If $m = n$ then the row space equals the column space.
   
   False. The dimensions are equal, but the spaces are not. A good example to look at is $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

6. True or false: The matrices $A$ and $-A$ share the same four subspaces.
   
   True, because whenever a vector $v$ is in a space, so is $-v$.

7. True or false: If $A$ and $B$ have the same four subspaces, then $A$ is a multiple of $B$.
   
   A good way to approach this question is to first try to convince yourself that it isn’t true – look for a counterexample. If $A$ is 3 by 3 and invertible, then its row and column space are both $\mathbb{R}^3$ and its nullspaces are $\{0\}$. If $B$ is any other invertible 3 by 3 matrix it will have the same four subspaces, and it may not be a multiple of $A$. So we answer “false”.

   It’s good to ask how we could truthfully complete the statement “If $A$ and $B$ have the same four subspaces, then ...”

8. If we exchange two rows of $A$, which subspaces stay the same?
   
   The row space and the nullspace stay the same.
9. Why can’t a vector \( \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \) be in the nullspace of \( A \) and also be a row of \( A \)?

Because if \( \mathbf{v} \) is the \( n \text{th} \) row of \( A \), the \( n \text{th} \) component of the vector \( A\mathbf{v} \) would be 14, not 0. The vector \( \mathbf{v} \) could not be a solution to \( A\mathbf{v} = \mathbf{0} \).

In fact, we will learn that the row space is perpendicular to the nullspace.