18.06SC Unit 1 Exam Solutions

1 (24 pts.) This question is about an m by n matrix A for which

$$Ax = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ has } \textbf{no solutions} \quad \text{and} \quad Ax = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ has } \textbf{exactly one solution}.$$

(a) Give all possible information about m and n and the rank r of A.

(b) Find all solutions to $Ax = 0$ and **explain your answer**.

(c) Write down an example of a matrix A that fits the description in part (a).

Solution.

(a) $Ax = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ has *one* solution $\Longrightarrow N(A) = \{0\}$ so $r = n$. (Also, $m = 3$ since $Ax \in \mathbb{R}^3$.)

$Ax = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ has no solution $\Longrightarrow C(A) \neq \mathbb{R}^3$, so $r < m$.

There are two possibilities: $\begin{matrix} m = 3 \\ r = n = 1 \end{matrix}$ and $\begin{matrix} m = 3 \\ r = n = 2 \end{matrix}$.

(b) Since $N(A) = \{0\}$ (because $Ax = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ has **1** solution), there is a unique solution to $Ax = 0$, which is clearly $x = 0$. (Can be either $x = \begin{bmatrix} 0 \end{bmatrix}$ or $x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ depending on if $n = 1$ or $n = 2$.)

(c) A could be $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ (many more possibilities).

2 (24 pts.) The 3 by 3 matrix A reduces to the identity matrix I by the following three row operations (in order):

E_{21} : Subtract 4 (row 1) from row 2.

E_{31} : Subtract 3 (row 1) from row 3.

E_{23} : Subtract row 3 from row 2.

(a) Write the inverse matrix A^{-1} in terms of the E's. **Then compute A^{-1}.**

(b) What is the original matrix A?

(c) What is the lower triangular factor L in $A = LU$?

Solution.

(a) Apply the three operations to I, i.e. $A^{-1} = E_{23}E_{31}E_{21}$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ -3 & 0 & 1 \end{bmatrix} = A^{-1}$$

(b) Apply the inverse operations in reverse order to I, i.e. $A = E_{21}^{-1}E_{31}^{-1}E_{23}^{-1}$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 3 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 1 \\ 3 & 0 & 1 \end{bmatrix} = A$$

$$\text{Check} \quad \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 1 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ -3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(c) $$L = \begin{bmatrix} 1 & & \\ 4 & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ 3 & & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}.$$

3 (28 pts.) This 3 by 4 matrix depends on c:

$$A = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 3 & c & 2 & 8 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

(a) *For each* c find a basis for the column space of A.

(b) *For each* c find a basis for the nullspace of A.

(c) *For each* c find the complete solution x to $Ax = \begin{bmatrix} 1 \\ c \\ 0 \end{bmatrix}$.

Solution.

(a) Elimination gives $\begin{bmatrix} \boxed{1} & 1 & 2 & 4 \\ 0 & c-3 & -4 & -4 \\ 0 & 0 & 2 & 2 \end{bmatrix}$ so there are two cases:

If $c \neq 3$, $c-3$ is a pivot and $U = \begin{bmatrix} \boxed{1} & 1 & 2 & 4 \\ 0 & \boxed{c-3} & -4 & -4 \\ 0 & 0 & \boxed{2} & 2 \end{bmatrix} \longrightarrow R = \begin{bmatrix} \boxed{1} & 0 & 0 & 2 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 1 \end{bmatrix}$

so a basis for $C(A)$ is the first three columns of A: $\left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ c \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \right\}$.

If $c = 3$, $c - 3 = 0$ and $U = \begin{bmatrix} \boxed{1} & 1 & 2 & 4 \\ 0 & 0 & \boxed{-4} & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow R = \begin{bmatrix} \boxed{1} & 1 & 0 & 2 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

so take the first and third columns of A as a basis for $C(A)$: $\left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \right\}$.

(b) If $c \neq 3$, the special solutions give $N(A) = \left\{ x_4 \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

If $c = 3$, the special solutions give $N(A) = \left\{ x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

(c) By inspection, $x_p = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ is one particular solution (other correct answers)

for $c \neq 3$, the complete solution is $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

for $c = 3$, the complete solution is $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

4 (24 pts.) (a) If A is a 3 by 5 matrix, what information do you have about the nullspace of A?

(b) Suppose row operations on A lead to this matrix $R = \texttt{rref}(A)$:

$$R = \begin{bmatrix} 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Write all known information about the columns of A.

(c) In the vector space M of all 3 by 3 matrices (you could call this a matrix space), what subspace S is spanned by all possible row reduced echelon forms R?

Solution.

(a) $N(A)$ has dimension *at least* 2 (and at most 5).

(b) (**7pts**) Columns 1, 4, 5 of A form a basis for $C(A)$.

($\approx$ **1pt**) Column 2 is $4 \times$ (Column 1); Column 3 is $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

(c) $A = \left\{ \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} \right\}$, the set of upper triangular matrices.

(A basis of six echelon forms is

$$\left\{ \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, \begin{bmatrix} 1 & & \\ & 1 & \\ & & \end{bmatrix}, \begin{bmatrix} 1 & & \\ & & 1 \\ & & \end{bmatrix}, \begin{bmatrix} 1 & & \\ & & \\ & & \end{bmatrix}, \begin{bmatrix} 1 & 1 & \\ & & \\ & & \end{bmatrix}, \begin{bmatrix} 1 & & 1 \\ & & \\ & & \end{bmatrix} \right\}.)$$