Exercises on matrix spaces; rank 1; small world graphs

**Problem 11.1:** [Optional] (3.5 #41. *Introduction to Linear Algebra*: Strang) Write the 3 by 3 identity matrix as a combination of the other five permutation matrices. Then show that those five matrices are linearly independent. (Assume a combination gives $c_1 P_1 + \cdots + c_5 P_5 = 0$ and check entries to prove $c_i$ is zero.) The five permutation matrices are a basis for the subspace of three by three matrices with row and column sums all equal.

**Problem 11.2:** (3.6 #31.) $M$ is the space of three by three matrices. Multiply each matrix $X$ in $M$ by:

$$A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Notice that $A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

a) Which matrices $X$ lead to $AX = 0$?

b) Which matrices have the form $AX$ for some matrix $X$?

c) Part (a) finds the “nullspace” of the operation $AX$ and part (b) finds the “column space.” What are the dimensions of those two subspaces of $M$? Why do the dimensions add to $(n - r) + r = 9$?