Cramer’s rule, inverse matrix, and volume

We know a formula for and some properties of the determinant. Now we see how the determinant can be used.

**Formula for** $A^{-1}$

We know:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$ 

Can we get a formula for the inverse of a 3 by 3 or $n$ by $n$ matrix? We expect that $\frac{1}{\det A}$ will be involved, as it is in the 2 by 2 example, and by looking at the cofactor matrix $\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$ we might guess that cofactors will be involved.

In fact:

$$A^{-1} = \frac{1}{\det A} C^T$$

where $C$ is the matrix of cofactors – please notice the transpose! Cofactors of row one of $A$ go into column 1 of $A^{-1}$, and then we divide by the determinant.

The determinant of $A$ involves products with $n$ terms and the cofactor matrix involves products of $n - 1$ terms. $A$ and $\frac{1}{\det A} C^T$ might cancel each other. This is much easier to see from our formula for the determinant than when using Gauss-Jordan elimination.

To more formally verify the formula, we’ll check that $AC^T = (\det A) I$.

$$AC^T = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & \cdots & C_{1n} \\ \vdots & \ddots & \vdots \\ C_{n1} & \cdots & C_{nn} \end{bmatrix}.$$ 

The entry in the first row and first column of the product matrix is:

$$\sum_{j=1}^{n} a_{1j}C_{1j} = \det A.$$ 

(This is just the cofactor formula for the determinant.) This happens for every entry on the diagonal of $AC^T$.

To finish proving that $AC^T = (\det A) I$, we just need to check that the off-diagonal entries of $AC^T$ are zero. In the two by two case, multiplying the entries in row 1 of $A$ by the entries in column 2 of $C^T$ gives $a(-b) + b(a) = 0$. This is the determinant of $A_s = \begin{bmatrix} a & b \\ a & b \end{bmatrix}$. In higher dimensions, the product of the first row of $A$ and the last column of $C^T$ equals the determinant of a matrix whose first and last rows are identical. This happens with all the off diagonal matrices, which confirms that $A^{-1} = \frac{1}{\det A} C^T$. 

1
This formula helps us answer questions about how the inverse changes when the matrix changes.

**Cramer’s Rule for** \( x = A^{-1}b \)

We know that if \( Ax = b \) and \( A \) is nonsingular, then \( x = A^{-1}b \). Applying the formula \( A^{-1} = C^T / \det A \) gives us:

\[
x = \frac{1}{\det A} C^T b.
\]

*Cramer’s rule* gives us another way of looking at this equation. To derive this rule we break \( x \) down into its components. Because the \( i \)’th component of \( C^T b \) is a sum of cofactors times some number, it is the determinant of some matrix \( B_j \).

\[
x_j = \frac{\det B_j}{\det A},
\]

where \( B_j \) is the matrix created by starting with \( A \) and then replacing column \( j \) with \( b \), so:

\[
B_1 = \begin{bmatrix} b & \text{last n-1 columns of } A \end{bmatrix}
\]

and

\[
B_n = \begin{bmatrix} \text{first n-1 columns of } A & b \end{bmatrix}.
\]

This agrees with our formula \( x_1 = \frac{\det B_1}{\det A} \). When taking the determinant of \( B_1 \) we get a sum whose first term is \( b_1 \) times the cofactor \( C_{11} \) of \( A \).

Computing inverses using Cramer’s rule is usually less efficient than using elimination.

\[| \det A | = \text{volume of box}\]

Claim: \( | \det A | \) is the volume of the box (*parallelepiped*) whose edges are the column vectors of \( A \). (We could equally well use the row vectors, forming a different box with the same volume.)

If \( A = I \), then the box is a unit cube and its volume is 1. Because this agrees with our claim, we can conclude that the volume obeys determinant property 1.

If \( A = Q \) is an orthogonal matrix then the box is a unit cube in a different orientation with volume \( 1 = | \det Q | \). (Because \( Q \) is an orthogonal matrix, \( Q^T Q = I \) and so \( \det Q = \pm 1 \).)

Swapping two columns of \( A \) does not change the volume of the box or (remembering that \( \det A = \det A^T \)) the absolute value of the determinant (property 2). If we show that the volume of the box also obeys property 3 we’ll have proven \( | \det A | \) equals the volume of the box.

2
Figure 1: The box whose edges are the column vectors of $A$.

If we double the length of one column of $A$, we double the volume of the box formed by its columns. Volume satisfies property 3(a).

Property 3(b) says that the determinant is linear in the rows of the matrix:

$$
\begin{vmatrix}
  a + a' & b + b' \\
  c & d
\end{vmatrix}
= \begin{vmatrix}
  a & b \\
  c & d
\end{vmatrix}
+ \begin{vmatrix}
  a' & b' \\
  c & d
\end{vmatrix}.
$$

Figure 2 illustrates why this should be true.

Figure 2: Volume obeys property 3(b).

Although it’s not needed for our proof, we can also see that determinants obey property 4. If two edges of a box are equal, the box flattens out and has no volume.
Important note: If you know the coordinates for the corners of a box, then computing the volume of the box is as easy as calculating a determinant. In particular, the area of a parallelogram with edges $\begin{bmatrix} a \\ b \end{bmatrix}$ and $\begin{bmatrix} c \\ d \end{bmatrix}$ is $ad - bc$.

The area of a triangle with edges $\begin{bmatrix} a \\ b \end{bmatrix}$ and $\begin{bmatrix} c \\ d \end{bmatrix}$ is half the area of that parallelogram, or $\frac{1}{2}(ad - bc)$. The area of a triangle with vertices at $(x_1, y_1)$, $(x_2, y_2)$ and $(x_3, y_3)$ is:

$$\frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$