Differential equations and $e^{At}$

The system of equations below describes how the values of variables $u_1$ and $u_2$ affect each other over time:

$$\frac{du_1}{dt} = -u_1 + 2u_2$$
$$\frac{du_2}{dt} = u_1 - 2u_2.$$

Just as we applied linear algebra to solve a difference equation, we can use it to solve this differential equation. For example, the initial condition $u_1 = 1$, $u_2 = 0$ can be written $u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Differential equations $\frac{du}{dt} = Au$

By looking at the equations above, we might guess that over time $u_1$ will decrease. We can get the same sort of information more safely by looking at the eigenvalues of the matrix $A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}$ of our system $\frac{du}{dt} = Au$. Because $A$ is singular and its trace is $-3$ we know that its eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = -3$. The solution will turn out to include $e^{-3t}$ and $e^{0t}$. As $t$ increases, $e^{-3t}$ vanishes and $e^{0t} = 1$ remains constant. Eigenvalues equal to zero have eigenvectors that are steady state solutions.

$x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector for which $Ax_1 = 0x_1$. To find an eigenvector corresponding to $\lambda_2 = -3$ we solve $(A - \lambda_2 I)x_2 = 0$:

$$\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} x_2 = 0$$

so $x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

and we can check that $Ax_2 = -3x_2$. The general solution to this system of differential equations will be:

$$u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2.$$

Is $e^{\lambda_1 t} x_1$ really a solution to $\frac{du}{dt} = Au$? To find out, plug in $u = e^{\lambda_1 t} x_1$:

$$\frac{du}{dt} = \lambda_1 e^{\lambda_1 t} x_1,$$

which agrees with:

$$Au = e^{\lambda_1 t} Ax_1 = \lambda_1 e^{\lambda_1 t} x_1.$$

The two “pure” terms $e^{\lambda_1 t} x_1$ and $e^{\lambda_2 t} x_2$ are analogous to the terms $\lambda_i^k x_i$ we saw in the solution $c_1 \lambda_1^1 x_1 + c_2 \lambda_2^1 x_2 + \cdots + c_n \lambda_n^1 x_n$ to the difference equation $u_{k+1} = Au_k$. 1
Plugging in the values of the eigenvectors, we get:

\[
\mathbf{u}(t) = c_1 e^{\lambda_1 t} \mathbf{x}_1 + c_2 e^{\lambda_2 t} \mathbf{x}_2 = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.
\]

We know \( \mathbf{u}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \), so at \( t = 0 \):

\[
\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}.
\]

\( c_1 = c_2 = 1/3 \) and \( \mathbf{u}(t) = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{3} e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \).

This tells us that the system starts with \( u_1 = 1 \) and \( u_2 = 0 \) but that as \( t \) approaches infinity, \( u_1 \) decays to 2/3 and \( u_2 \) increases to 1/3. This might describe stuff moving from \( u_1 \) to \( u_2 \).

The steady state of this system is \( \mathbf{u}(\infty) = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} \).

**Stability**

Not all systems have a steady state. The eigenvalues of \( \mathbf{A} \) will tell us what sort of solutions to expect:

1. **Stability:** \( \mathbf{u}(t) \to 0 \) when \( \text{Re}(\lambda) < 0 \).
2. **Steady state:** One eigenvalue is 0 and all other eigenvalues have negative real part.
3. **Blow up:** if \( \text{Re}(\lambda) > 0 \) for any eigenvalue \( \lambda \).

If a two by two matrix \( \mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) has two eigenvalues with negative real part, its trace \( a + d \) is negative. The converse is not true: \( \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} \) has negative trace but one of its eigenvalues is 1 and \( e^{1t} \) blows up. If \( \mathbf{A} \) has a positive determinant and negative trace then the corresponding solutions must be stable.

**Applying S**

The final step of our solution to the system \( \frac{\text{d}\mathbf{u}}{\text{d}t} = \mathbf{A}\mathbf{u} \) was to solve:

\[
c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.
\]

In matrix form:

\[
\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.
\]
or \( Sc = u(0) \), where \( S \) is the eigenvector matrix. The components of \( c \) determine the contribution from each pure exponential solution, based on the initial conditions of the system.

In the equation \( \frac{du}{dt} = Au \), the matrix \( A \) couples the pure solutions. We set \( u = Sv \), where \( S \) is the matrix of eigenvectors of \( A \), to get:

\[
S \frac{dv}{dt} = ASv
\]

or:

\[
\frac{dv}{dt} = S^{-1}ASv = \Lambda v.
\]

This diagonalizes the system: \( \frac{dv}{dt} = \Lambda v \). The general solution is then:

\[
v(t) = e^{At}v(0), \quad \text{and} \quad u(t) = Se^{At}S^{-1}v(0) = e^{At}u(0).
\]

**Matrix exponential \( e^{At} \)**

What does \( e^{At} \) mean if \( A \) is a matrix? We know that for a real number \( x \),

\[
e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots.
\]

We can use the same formula to define \( e^{At} \):

\[
e^{At} = I + At + \frac{(At)^2}{2} + \frac{(At)^3}{6} + \cdots.
\]

Similarly, if the eigenvalues of \( At \) are small, we can use the geometric series

\[
\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \text{ to estimate } (I - At)^{-1} = I + At + (At)^2 + (At)^3 + \cdots.
\]

We’ve said that \( e^{At} = Se^{At}S^{-1} \). If \( A \) has \( n \) independent eigenvectors we can prove this from the definition of \( e^{At} \) by using the formula \( A = S\Lambda S^{-1} \):

\[
e^{At} = I + At + \frac{(At)^2}{2} + \frac{(At)^3}{6} + \cdots
= SS^{-1} + S\Lambda S^{-1}t + \frac{S\Lambda^2 S^{-1}t^2}{2} + \frac{S\Lambda^3 S^{-1}t^3}{6} + \cdots
= Se^{At}S^{-1}.
\]

It’s impractical to add up infinitely many matrices. Fortunately, there is an easier way to compute \( e^{At} \). Remember that:

\[
\Lambda = \begin{bmatrix}
\lambda_1 & 0 & \cdots & 0 \\
0 & \lambda_2 & 0 & \cdots \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & \lambda_n
\end{bmatrix}
\]
When we plug this in to our formula for $e^{At}$ we find that:

$$e^{A t} = \begin{bmatrix}
e^{\lambda_1 t} & 0 & \cdots & 0 \\
0 & e^{\lambda_2 t} & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & e^{\lambda_n t}
\end{bmatrix}. \tag{20}$$

This is another way to see the relationship between the stability of $u(t) = Se^{At}S^{-1}v(0)$ and the eigenvalues of $A$.

**Second order**

We can change the second order equation $y'' + by' + ky = 0$ into a two by two first order system using a method similar to the one we used to find a formula for the Fibonacci numbers. If $u = \begin{bmatrix} y' \\ y \end{bmatrix}$, then

$$u' = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix}
-b & -k \\
1 & 0
\end{bmatrix} \begin{bmatrix} y' \\ y \end{bmatrix}. \tag{21}$$

We could use the methods we just learned to solve this system, and that would give us a solution to the second order scalar equation we started with.

If we start with a $k$th order equation we get a $k$ by $k$ matrix with coefficients of the equation in the first row and 1’s on a diagonal below that; the rest of the entries are 0.