Exercises on Markov matrices; Fourier series

Problem 24.1: (6.4 #7. Introduction to Linear Algebra: Strang)

a) Find a symmetric matrix \( \begin{bmatrix} 1 & b \\ b & 1 \end{bmatrix} \) that has a negative eigenvalue.

b) How do you know it must have a negative pivot?

c) How do you know it can’t have two negative eigenvalues?

Solution:

a) The eigenvalues of that matrix are \( 1 \pm b \). If \( b > 1 \) or \( b < -1 \) the matrix has a negative eigenvalue.

b) The pivots have the same signs as the eigenvalues. If the matrix has a negative eigenvalue, then it must have a negative pivot.

c) To obtain one negative eigenvalue, we choose either \( b > 1 \) or \( b < -1 \) (as stated in part (a)). If we choose \( b > 1 \), then \( \lambda_1 = 1 + b \) will be positive while \( \lambda_2 = 1 - b \) will be negative. Alternatively, if we choose \( b < -1 \), then \( \lambda_1 = 1 + b \) will be negative while \( \lambda_2 = 1 - b \) will be positive. Therefore this matrix cannot have two negative eigenvalues.

Problem 24.2: (6.4 #23.) Which of these classes of matrices do \( A \) and \( B \) belong to: invertible, orthogonal, projection, permutation, diagonalizable, Markov?

\[
A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad B = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.
\]

Which of these factorizations are possible for \( A \) and \( B \): \( LU \), \( QR \), \( SAS^{-1} \), or \( QΛQ^T \)?
Solution:

a) For $A$:

- $\det A = -1 \neq 0$. $A$ is invertible.
- $AA^T = I$. $A$ is orthogonal.
- $A^2 = I \neq A$. $A$ is not a projection.
- $A$ has one 1 in each row and column with 0’s elsewhere. $A$ is a permutation.
- $A = A^T$, so $A$ is symmetric. $A$ is diagonalizable.
- Each column of $A$ sums to one. $A$ is Markov.
- $A = LU$ is not possible because $A_{11} = 0$. $QR$ is possible because $A$ has independent columns, $SAS^{-1}$ is possible because it is diagonalizable, and $QAQ^T$ is possible because it is symmetric.

b) For $B$:

- $\det B = 0$. $B$ is not invertible.
- $BB^T \neq I$. $B$ is not orthogonal.
- $B^2 = B$. $B$ is a projection.
- $B$ does not have one 1 in each row and each column, with 0’s elsewhere. $B$ is not a permutation.
- $B = B^T$ so $B$ is symmetric. $B$ is diagonalizable.
- Each column of $B$ sums to one. $B$ is Markov.
- $B = LU$ is possible but $U$ only contains one nonzero pivot. $QR$ is impossible because $B$ has dependent columns, $SAS^{-1}$ is possible because it is diagonalizable, and $QAQ^T$ is possible because it is symmetric.

Problem 24.3: (8.3 #11.) Complete $A$ to a Markov matrix and find the steady state eigenvector. When $A$ is a symmetric Markov matrix, why is $x_1 = (1, \ldots, 1)$ its steady state?

$$A = \begin{bmatrix} .7 & .1 & .2 \\ .1 & .6 & .3 \\ \_ & \_ & \_ \end{bmatrix}.$$ 

Solution: Matrix $A$ becomes:

$$A = \begin{bmatrix} .7 & .1 & .2 \\ .1 & .6 & .3 \\ .2 & .3 & .5 \end{bmatrix},$$
with steady state vector (1,1,1). When $A$ is a symmetric Markov matrix, the elements of each row sum to one. The elements of each row of $A - I$ then sum to zero. Since the steady state vector $x$ is the eigenvector associated with eigenvalue $\lambda = 1$, we solve $(A - \lambda I)x = (A - I)x = 0$ to get $x = (1, \ldots, 1)$. 