Exercises on orthogonal matrices and Gram-Schmidt

Problem 17.1: (4.4 #10.b Introduction to Linear Algebra: Strang)
Orthogonal vectors are automatically linearly independent.
Matrix Proof: Show that $Qx = 0$ implies $x = 0$. Since $Q$ may be rectangular, you can use $Q^T$ but not $Q^{-1}$.

Solution: By definition, $Q$ is a matrix whose columns are orthonormal, and so we know that $Q^TQ = I$ (where $Q$ may be rectangular). Then:

$$Qx = 0 \implies Q^TQx = 0 \implies Ix = 0 \implies x = 0.$$  

Thus the nullspace of $Q$ is the zero vector, and so the columns of $Q$ are linearly independent. There are no non-zero linear combinations of the columns that equal the zero vector. Thus, orthonormal vectors are automatically linearly independent.

Problem 17.2: (4.4 #18) Given the vectors $a$, $b$ and $c$ listed below, use the Gram-Schmidt process to find orthogonal vectors $A$, $B$, and $C$ that span the same space.

$$a = (1, -1, 0, 0), \quad b = (0, 1, -1, 0), \quad c = (0, 0, 1, -1).$$

Show that $\{A, B, C\}$ and $\{a, b, c\}$ are bases for the space of vectors perpendicular to $d = (1, 1, 1, 1)$.

Solution: We apply Gram-Schmidt to $a$, $b$, $c$. First, we set

$$A = a = (1, -1, 0, 0).$$

Next we find $B$:

$$B = b - \frac{A^Tb}{A^TA}A = (0, 1, -1, 0) + \frac{1}{2}(1, -1, 0, 0) = \left( \frac{1}{2}, \frac{1}{2}, 1, 0 \right).$$

And then we find $C$:

$$C = c - \frac{A^Tc}{A^TA}A - \frac{B^Tc}{B^TB}B = (0, 0, 1, -1) + \frac{2}{3}\left( \frac{1}{3}, \frac{1}{3}, -1, 0 \right) = \left( \frac{1}{3}, \frac{1}{3}, 1, -1 \right).$$
We know from the first problem that the elements of the set \{A, B, C\} are linearly independent, and each vector is orthogonal to (1,1,1,1). The space of vectors perpendicular to \( \mathbf{d} \) is three dimensional (since the row space of \((1, 1, 1, 1)\) is one-dimensional, and the number of dimensions of the row space added to the number of dimensions of the nullspace add to 4). Therefore \{A, B, C\} forms a basis for the space of vectors perpendicular to \( \mathbf{d} \).

Similarly, \{a, b, c\} is a basis for the space of vectors perpendicular to \( \mathbf{d} \) because the vectors are linearly independent, orthogonal to \((1,1,1,1)\), and because there are three of them.
18.06SC Linear Algebra
Fall 2011

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