Hi. Welcome to recitation.

Today's this problem is about change of basis. It says the vector space of polynomials in x of degree up to 2 has a basis 1, x, and x squared. That's the obvious basis that you would write for that vector space.

But today we're going to consider another basis, w1, w2, and w3. And we don't know what w1, w2, and w3 are explicitly. What we know is that their values at x equals minus 1, 0, and 1 are given by this table here. So there are 1 0 0, 0 1 0, and 0 0 1.

We're asked to do the following. We're asked to express this polynomial. So y of x is minus x plus 5 in this basis, w1, w2, and w3. We're asked to find the change of basis matrices between these two bases, 1, x, x squared, and w1, w2, w3. And finally, we're asked to find the matrix of taking derivatives, which is a linear map on this space, in both of these basis.

And let me give you an extra level of challenge, which is to try to do as much of this as possible without explicitly finding w1, w2, and w3. I'll let you think about the problem and then you can come back and compare your solution with mine.

Hi. Welcome back. So to start with the problem a, we need to find coefficients alpha, theta, and gamma. So that y of x is expressed through, with this coefficient, in this new basis, w1, w2, and w3.

now, one way to do that would be to look at this table of values, explicitly find w1, w2, and w3, so a quadratic polynomial is-- all the information we need about it is in values at three points. So say w1 is a1 plus b times x plus c times x squared. Find a, b, and c. Find w1, w2, w3 explicitly, and then go back to this system and try to find alpha, beta and gamma.

However, there's a trick. Let's try to see if we can do it without finding w1, w2, and
w3 explicitly. So let me try to see what are the values of y at these points. So y is
minus x plus 5. So the values are 6, 5, and 4.

And let me, instead of considering this equation, let me evaluate it at x is minus 1, x
is 0, and x is 1. What I get through this is that w is what is minus 1. Actually, let me
write this-- I get w with minus 1, which is a number, equals alpha times w1 is minus
1, which is a number, plus beta times w2 is minus 1, plus gamma times w3 at minus
1.

And similarly at 0, and similarly at 1. And now let me think of this as a linear system
that has unknowns alpha, beta, and gamma coefficients, these values here at minus
1, 0, and 1. And the right-hand side, well what's written here on the left-hand side. y
at minus 1, y at 0, and y at 1.

If you write this in a matrix and read off coefficients from there, you get the following
system. So this is the matrix of the system read off from here. These are the
unknowns. And these are the values of the right-hand side. And I hope you'll agree
that this is a very easy system to solve. We just get alpha is 6, beta is 5, and
gamma is 4.

So the solution to the first part is y equals 6w1 plus 5w2 plus 4w3. And let's notice
another thing. No matter what values we put here, this matrix is always going to stay
the same. It's only the right-hand side that's going to change. So if we're given any
other polynomial now to express in a basis w1, w2, and w3, we don't have to do any
thinking. We don't have to do any computations. What we do is go back to our table
at the beginning and just read off, let's go back to the table, and just read these
values. So in this case, y is 6 times w1, 5 times w2, and 3 times w3.

And that's already a hint to solving the b part, which is find the change of basis
matrices between 1, x, x squared and w1, w2, w3. Change of basis matrices means
expressing one basis in terms of another. So as a part of the problem, we will have
to express 1, x, and x squared in terms of w1, w2, w3. So let's just find their values
at these three points.
One is a constant, it just takes value 1 everywhere. \( x \) takes value minus 1 at minus 1, 0 at 0, and 1 at 1. And \( x^2 \) takes values 0 and 1 at minus 1 and 1. And from this we can already conclude the part b here, we can conclude that 1 equals \( w_1 + w_2 + w_3 \). That \( x \) equals minus \( w_1 + w_3 \). And that \( x^2 \) equals \( w_1 + w_3 \).

And from this, we can immediately write one change of basis matrix. Namely, since we know how to express 1, \( x \), and \( x^2 \) in terms of \( w_1 \), \( w_2 \), and \( w_3 \), we can just copy this information over to this matrix, getting 1 1 1, minus 1 0 1, and 1 0 1.

So which matrix is this? This is a matrix, so we have 1, \( x \), and \( x^2 \) expressed in terms of \( w_1 \), \( w_2 \), and \( w_3 \). So if we feed this matrix something expressed in the basis 1, \( x \), and \( x^2 \), say \( a \), \( b \), and \( c \), what is going to throw out is the same polynomial expressed in this basis here, \( w_1 \), \( w_2 \), and \( w_3 \).

So I'm going to just write that this is a matrix of this basis change.

How do we get the other one? Well very easy. We know it's just the inverse of \( A \). So this is going to be the matrix that takes something written in this basis and transfers it to this basis. I'm not going to calculate the inverse of a matrix in front of you. Instead I'm going to consult my oracle. Sorry about that. And my oracle says that the inverse should be 0 1 0, minus 1/2 0 1/2, and 1/2 minus 1 1/2. And that solves the b part.

Let's go into the c part. The c part required us to find a matrix of taking derivatives, which is a linear map in the space of polynomials, in both of these basis. So let's first do the 1, \( x \), \( x^2 \) basis because that one's easier. I'm going to call it \( D_x \). So I'm going to work in basis 1, \( x \), \( x^2 \). And what I want to express is the transformation of taking derivatives.

So here I'm going to write the vector to which taking derivatives maps the polynomial 1, which is 0. And that this expressed in the basis 1, \( x \), \( x^2 \). In the second column I'm going to write \( x \) prime, the vector to which \( D_x \) sends to vector \( x \), and that's equal to 1, which expressed in this basis is 1 0 0. And here I'm going to
write x squared prime, which is 2x, which expressed in this basis is just x.

That one was easy. For the other one, well we could calculate w1, w2, w3 explicitly, take the derivatives, go back to the table and repeat the procedure that we did already. So expressing these derivatives in terms of w1, w2, w3, and that's a lot of work. But we pretty much already did most of this work. So we know how to take derivatives in this basis, and we know how to go between these two basis.

So if we want to take a derivative of something written in the basis w1, w2, w3, well let's first write this something in basis 1, x, x squared. Then let's take a derivative of it. And then let's write it back in the original basis that we want. So it's multiplication of three matrices. We have all three-- matrix multiplication is easy. And my oracle, again, says that this should be minus 3 over 2, 2, minus 1 over 2, minus 1 over 2, 0, 1 over 2, and 1 over 2, minus 2, 3 over 2. And that solves the problem.

Now, one thing that I would want to discuss in the end is how did you do with respect to my challenge, which was let's try to do as much of it as possible without finding w1, w2, and w3 explicitly. And it seems like we did really well. There's nowhere on the board written w1 equals, w2 equals, w3 equals.

But is it really so? It's not. We calculated the matrix of A inverse here. And what this really means is that w1, w2, and w3 written in the basis 1, x, and x squared are as follows. w1 is minus 1/2 x plus 1/2 x squared. w2 is 1 minus x squared. And w3 is 1/2 x plus 1/2 x squared. So you can check your work with the help of this matrix in case you did find w1, w2, and w3 explicitly. And that's all I wanted to say today.