Exercises on positive definite matrices and minima

Problem 27.1: (6.5 #33. Introduction to Linear Algebra: Strang) When $A$ and $B$ are symmetric positive definite, $AB$ might not even be symmetric, but its eigenvalues are still positive. Start from $ABx = \lambda x$ and take dot products with $Bx$. Then prove $\lambda > 0$.

Solution:

$$ABx = \lambda x$$

$$(ABx)^TBx = (\lambda x)^TBx$$

$$(Bx)^TA^TBx = \lambda x^TBx$$

$$(Bx)^TA(Bx) = \lambda(x^TBx).$$

where $A^T = A$ because $A$ is symmetric. Since $A$ is positive definite we know $(Bx)^TA(Bx) > 0$, and since $B$ is positive definite $x^TBx > 0$. Hence, $\lambda$ must be positive as well.

Problem 27.2: Find the quadratic form associated with the matrix \[ \begin{bmatrix} 1 & 5 \\ 7 & 9 \end{bmatrix} \]. Is this function $f(x,y)$ always positive, always negative, or sometimes positive and sometimes negative?

Solution: To find the quadratic form, compute $x^TAx$:

$$f(x,y) = \begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 1 & 5 \\ 7 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = x(x + 5y) + y(7x + 9y) = x^2 + 12xy + 9y^2.$$ 

This expression can be positive, e.g. when $y = 0$ and $x \neq 0$.

The expression will sometimes be negative because $A$ is not positive definite. For instance, $f(2,-2) = -8$. Thus the quadratic form associated with the matrix $A$ is sometimes positive and sometimes negative. Another way to reach this conclusion is to note that $\det A = -26$ is negative and so $A$ is not positive definite.