Exercises on singular value decomposition

Problem 29.1:  (Based on 6.7 #4. Introduction to Linear Algebra: Strang) Verify that if we compute the singular value decomposition \( A = U \Sigma V^T \) of the Fibonacci matrix \( A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \),
\[
\Sigma = \begin{bmatrix}
\frac{1+\sqrt{5}}{2} & 0 \\
0 & \frac{\sqrt{5}-1}{2}
\end{bmatrix}.
\]

Solution:
\[
A^T A = A A^T = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.
\]
The eigenvalues of this matrix are the roots of \( x^2 - 3x + 1 \), which are \( \frac{3 \pm \sqrt{5}}{2} \). Thus we have:
\[
\sigma_1^2 = \frac{3 + \sqrt{5}}{2} \quad \text{and} \quad \sigma_2^2 = \frac{3 - \sqrt{5}}{2}.
\]

To check that \( \Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \), we will square the entries of the matrix \( \Sigma \) given above.
\[
\left( \frac{1 + \sqrt{5}}{2} \right)^2 = \frac{1 + 2\sqrt{5} + 5}{4} = \frac{3 + \sqrt{5}}{2}. \quad \checkmark
\]
\[
\left( \frac{\sqrt{5} - 1}{2} \right)^2 = \frac{5 - 2\sqrt{5} + 1}{4} = \frac{3 - \sqrt{5}}{2}. \quad \checkmark
\]

Problem 29.2:  (6.7 #11.) Suppose \( A \) has orthogonal columns \( \mathbf{w}_1, \mathbf{w}_2, ..., \mathbf{w}_n \) of lengths \( \sigma_1, \sigma_2, ..., \sigma_n \). Calculate \( A^T A \). What are \( U, \Sigma, \) and \( V \) in the SVD?
Solution: Since the columns of $A$ are orthogonal, $A^T A$ is a diagonal matrix with entries $\sigma_1^2, ..., \sigma_n^2$. Since $A^T A = V \Sigma^2 V^T$, we find that $\Sigma^2$ is the matrix with diagonal entries $\sigma_1^2, ..., \sigma_n^2$ and thus that $\Sigma$ is the matrix with diagonal entries $\sigma_1, ..., \sigma_n$.

Referring again to the equation $A^T A = V \Sigma^2 V^T$, we conclude also that $V = I$.

The equation $A = U \Sigma V^T$ then tells us that $U$ must be the matrix whose columns are $\frac{1}{\sigma_i} w_i$. 