Singular value decomposition

The singular value decomposition of a matrix is usually referred to as the SVD. This is the final and best factorization of a matrix:

\[ A = U \Sigma V^T \]

where \( U \) is orthogonal, \( \Sigma \) is diagonal, and \( V \) is orthogonal.

In the decomposition \( A = U \Sigma V^T \), \( A \) can be any matrix. We know that if \( A \) is symmetric positive definite its eigenvectors are orthogonal and we can write \( A = Q \Lambda Q^T \). This is a special case of a SVD, with \( U = V = Q \). For more general \( A \), the SVD requires two different matrices \( U \) and \( V \).

We’ve also learned how to write \( A = S \Lambda S^{-1} \), where \( S \) is the matrix of \( n \) distinct eigenvectors of \( A \). However, \( S \) may not be orthogonal; the matrices \( U \) and \( V \) in the SVD will be.

How it works

We can think of \( A \) as a linear transformation taking a vector \( v_1 \) in its row space to a vector \( u_1 = Av_1 \) in its column space. The SVD arises from finding an orthogonal basis for the row space that gets transformed into an orthogonal basis for the column space: \( Av_i = \sigma_i u_i \).

It’s not hard to find an orthogonal basis for the row space – the Gram-Schmidt process gives us one right away. But in general, there’s no reason to expect \( A \) to transform that basis to another orthogonal basis.

You may be wondering about the vectors in the nullspaces of \( A \) and \( A^T \). These are no problem – zeros on the diagonal of \( \Sigma \) will take care of them.

Matrix language

The heart of the problem is to find an orthonormal basis \( v_1, v_2, \ldots, v_r \) for the row space of \( A \) for which

\[ A \begin{bmatrix} v_1 & v_2 & \cdots & v_r \end{bmatrix} = \begin{bmatrix} \sigma_1 u_1 & \sigma_2 u_2 & \cdots & \sigma_r u_r \end{bmatrix} \]

\[ = \begin{bmatrix} u_1 & u_2 & \cdots & u_r \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_r \end{bmatrix} \]

with \( u_1, u_2, \ldots, u_r \) an orthonormal basis for the column space of \( A \). Once we add in the nullspaces, this equation will become \( AV = U \Sigma \). (We can complete the orthonormal bases \( v_1, \ldots, v_r \) and \( u_1, \ldots, u_r \) to orthonormal bases for the entire space any way we want. Since \( v_{r+1}, \ldots, v_n \) will be in the nullspace of \( A \), the diagonal entries \( \sigma_{r+1}, \ldots, \sigma_n \) will be 0.)

The columns of \( U \) and \( V \) are bases for the row and column spaces, respectively. Usually \( U \neq V \), but if \( A \) is positive definite we can use the same basis for its row and column space!
Calculation

Suppose $A$ is the invertible matrix $\begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$. We want to find vectors $v_1$ and $v_2$ in the row space $\mathbb{R}^2$, $u_1$ and $u_2$ in the column space $\mathbb{R}^2$, and positive numbers $\sigma_1$ and $\sigma_2$ so that the $v_i$ are orthonormal, the $u_i$ are orthonormal, and the $\sigma_i$ are the scaling factors for which $A v_i = \sigma_i u_i$.

This is a big step toward finding orthonormal matrices $V$ and $U$ and a diagonal matrix $\Sigma$ for which:

$$AV = U \Sigma.$$

Since $V$ is orthogonal, we can multiply both sides by $V^{-1} = V^T$ to get:

$$A = U \Sigma V^T.$$

Rather than solving for $U$, $V$ and $\Sigma$ simultaneously, we multiply both sides by $A^T V \Sigma^T U^T$ to get:

$$A^T A = V \Sigma U^{-1} U \Sigma V^T$$

$$= V \Sigma^2 V^T$$

$$= V \begin{bmatrix} \sigma_1^2 & 0 & \cdots \\ 0 & \sigma_2^2 & \cdots \\ \vdots & \vdots & \ddots \\ 0 & 0 & \cdots & \sigma_n^2 \end{bmatrix} V^T.$$

This is in the form $Q \Lambda Q^T$; we can now find $V$ by diagonalizing the symmetric positive definite (or semidefinite) matrix $A^T A$. The columns of $V$ are eigenvectors of $A^T A$ and the eigenvalues of $A^T A$ are the values $\sigma_i^2$. (We choose $\sigma_i$ to be the positive square root of $\lambda_i$.)

To find $U$, we do the same thing with $A A^T$.

SVD example

We return to our matrix $A = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$. We start by computing

$$A^T A = \begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix}.$$
Two orthogonal eigenvectors of $A^T A$ are $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$. To get an orthonormal basis, let $v_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$. These have eigenvalues $\sigma_1^2 = 32$ and $\sigma_2^2 = 18$. We now have:

$$
\begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} U = \begin{bmatrix} \sigma_1 \\ 0 \\ 0 \end{bmatrix} V^T
$$

$$
A U \Sigma V^T = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 4 \sqrt{2} & 0 \\ 0 & 3 \sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}.
$$

We could solve this for $U$, but for practice we’ll find $U$ by finding orthonormal eigenvectors $u_1$ and $u_2$ for $A A^T = U \Sigma^2 U^T$. We have:

$$
A A^T = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}
$$

$$
= \begin{bmatrix} 32 & 0 \\ 0 & 18 \end{bmatrix}.
$$

Luckily, $A A^T$ happens to be diagonal. It’s tempting to let $u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $u_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, as Professor Strang did in the lecture, but because $A v_2 = \begin{bmatrix} 0 \\ -3 \sqrt{2} \end{bmatrix}$ we instead have $u_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Note that this also gives us a chance to double check our calculation of $\sigma_1$ and $\sigma_2$.

Thus, the SVD of $A$ is:

$$
\begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} U = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix} V^T
$$

$$
= \begin{bmatrix} 4 \sqrt{2} & 0 \\ 0 & 3 \sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}.
$$

**Example with a nullspace**

Now let $A = \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}$. This has a one dimensional nullspace and one dimensional row and column spaces.

The row space of $A$ consists of the multiples of $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$. The column space of $A$ is made up of multiples of $\begin{bmatrix} 4 \\ 8 \end{bmatrix}$. The nullspace and left nullspace are perpendicular to the row and column spaces, respectively.

Unit basis vectors of the row and column spaces are $v_1 = \begin{bmatrix} .8 \\ .6 \end{bmatrix}$ and $u_1 = \begin{bmatrix} .5 \\ .3 \end{bmatrix}$.
\[
\begin{bmatrix}
1/\sqrt{5} \\
2/\sqrt{5}
\end{bmatrix}.
\]
To compute \(\sigma_1\) we find the nonzero eigenvalue of \(A^T A\).

\[
A^T A = \begin{bmatrix} 4 & 8 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}
= \begin{bmatrix} 80 & 60 \\ 60 & 45 \end{bmatrix}.
\]

Because this is a rank 1 matrix, one eigenvalue must be 0. The other must equal the trace, so \(\sigma_1^2 = 125\). After finding unit vectors perpendicular to \(u_1\) and \(v_1\) (basis vectors for the left nullspace and nullspace, respectively) we see that the SVD of \(A\) is:

\[
\begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} \sqrt{125} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} .8 & .6 \\ .6 & -.8 \end{bmatrix}.
\]

The singular value decomposition combines topics in linear algebra ranging from positive definite matrices to the four fundamental subspaces.

\(v_1, v_2, \ldots v_r\) is an orthonormal basis for the row space.

\(u_1, u_2, \ldots u_r\) is an orthonormal basis for the column space.

\(v_{r+1}, \ldots v_n\) is an orthonormal basis for the nullspace.

\(u_{r+1}, \ldots u_m\) is an orthonormal basis for the left nullspace.

These are the “right” bases to use, because \(A v_i = \sigma_i u_i\).