SOLUTION SET I FOR 18.075–FALL 2004

10. Functions of a Complex Variable

10.1. Introduction. The Complex Variable.

3. Establish the following results:
   (a) $\text{Re}(z_1 + z_2) = \text{Re}(z_1) + \text{Re}(z_2)$, but $\text{Re}(z_1z_2) \neq \text{Re}(z_1)\text{Re}(z_2)$ in general;
   (b) $\text{Im}(z_1 + z_2) = \text{Im}(z_1) + \text{Im}(z_2)$, but $\text{Im}(z_1z_2) \neq \text{Im}(z_1)\text{Im}(z_2)$ in general;
   (c) $|z_1z_2| = |z_1||z_2|$, but $|z_1 + z_2| \neq |z_1| + |z_2|$ in general;
   (d) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ and $\overline{z_1z_2} = \overline{z_1}\overline{z_2}$.

Solution. (a) We want to show that $\text{Re}(z_1 + z_2) = \text{Re}(z_1) + \text{Re}(z_2)$. Let $z_1 = a_1 + ib_1$, $z_2 = a_2 + ib_2$, then

$$z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2).$$

Hence

$$\text{Re}(z_1 + z_2) = a_1 + a_2$$

and clearly

$$\text{Re}(z_1) + \text{Re}(z_2) = a_1 + a_2.$$ 

Let us show that in general $\text{Re}(z_1z_2) \neq \text{Re}(z_1)\text{Re}(z_2)$. We have

$$z_1z_2 = (a_1 + ib_1)(a_2 + ib_2) = (a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1),$$

therefore

$$\text{Re}(z_1z_2) = a_1a_2 - b_1b_2.$$ 

On the other hand

$$\text{Re}(z_1)\text{Re}(z_2) = a_1a_2 \neq a_1a_2 - b_1b_2$$

in general.

(b) We want to show that $\text{Im}(z_1 + z_2) = \text{Im}(z_1) + \text{Im}(z_2)$. From part (a) we have that

$$z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2).$$

Hence

$$\text{Im}(z_1 + z_2) = b_1 + b_2$$

and clearly

$$\text{Im}(z_1) + \text{Im}(z_2) = b_1 + b_2.$$ 

Let us show that in general $\text{Im}(z_1z_2) \neq \text{Im}(z_1)\text{Im}(z_2)$. We have

$$z_1z_2 = (a_1 + ib_1)(a_2 + ib_2) = (a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1),$$

therefore

$$\text{Im}(z_1z_2) = a_1b_2 + a_2b_1.$$

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On the other hand

\[ \text{Im}(z_1)\text{Im}(z_2) = b_1b_2 \neq a_1b_2 + a_2b_1 \]

in general.

(c) We want to show that \( |z_1z_2| = |z_1||z_2| \). From part (a) we have

\[ z_1z_2 = (a_1 + ib_1)(a_2 + ib_2) = (a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1). \]

Hence

\[
|z_1z_2| = \sqrt{(a_1a_2 - b_1b_2)^2 + (a_1b_2 + a_2b_1)^2} \\
= \sqrt{(a_1^2a_2^2 - 2a_1a_2b_1b_2 + b_1^2b_2^2) + (a_1^2b_2^2 + 2a_1b_2a_2b_1 + a_2^2b_1^2)} \\
= \sqrt{a_1^2a_2^2 + b_1^2b_2^2 + a_2^2b_1^2 + a_1^2b_2^2 + a_2^2b_1^2}.
\]

On the other hand \( |z_1| = \sqrt{a_1^2 + b_1^2} \) and \( |z_2| = \sqrt{a_2^2 + b_2^2} \). Therefore

\[ |z_1||z_2| = (\sqrt{a_1^2 + b_1^2})(\sqrt{a_2^2 + b_2^2}) = \sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)} = \sqrt{a_1^2a_2^2 + b_1^2b_2^2 + a_2^2b_1^2 + a_1^2b_2^2} \]

which is equal to \( |z_1z_2| \).

Let us show that \( |z_1 + z_2| \neq |z_1| + |z_2| \) in general. From part (a) we have

\[ z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2). \]

Hence

\[ |z_1 + z_2| = \sqrt{(a_1 + a_2)^2 + (b_1 + b_2)^2}. \]

On the other hand

\[ |z_1| + |z_2| = \sqrt{(a_1 + a_2)^2 + (b_1 + b_2)^2} \]

and

\[ \sqrt{(a_1 + a_2)^2 + (b_1 + b_2)^2} \neq \sqrt{(a_1 + a_2)^2 + (b_1 + b_2)^2} \]

in general (choose for example \( a_1 = 1, b_1 = 0, a_2 = 0, b_2 = 1 \)).

(d) We want to show that \( \overline{z_1 + z_2} = \overline{z_1} + \overline{z_2} \). From part (a) we have

\[ z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2) \]

then

\[ \overline{z_1 + z_2} = (a_1 + a_2) - i(b_1 + b_2). \]

On the other hand

\[ \overline{z_1 + z_2} = (a_1 - ib_1) + (a_2 - ib_2) = (a_1 + a_2) - i(b_1 + b_2) \]

which is equal to \( \overline{z_1 + z_2} \).

Let us show that \( \overline{z_1z_2} = \overline{z_1}\overline{z_2} \). From part (a) we have that

\[ z_1z_2 = (a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1). \]

Hence

\[ \overline{z_1z_2} = (a_1a_2 - b_1b_2) - i(a_1b_2 + a_2b_1). \]

On the other hand,

\[ \overline{z_1z_2} = (a_1 - ib_1)(a_2 - ib_2) = (a_1a_2 - b_1b_2) - i(a_1b_2 + a_2b_1) \]

which is equal to \( \overline{z_1z_2} \).
4. Establish the following results:
(a) $z + \overline{z} = 2\text{Re}z$,
(b) $z - \overline{z} = 2i\text{Im}z$,
(c) $z_1\overline{z_2} + \overline{z_1}z_2 = 2(z_1\overline{z_2}) = 2\text{Re}(z_1\overline{z_2})$,
(d) $\text{Re}z \leq |z|$,  
(e) $|z| \leq |\text{Re}z|$,  
(f) $|z_1\overline{z_2} + \overline{z_1}z_2| \leq 2|z_1z_2|$,  
(g) $(|z_1| - |z_2|)^2 \leq |z_1 + z_2|^2 \leq (|z_1| + |z_2|)^2$. [Use part (f).]

Solution.
(a) Let $z = a + ib$, then

$$z + \overline{z} = (a + ib) + (a - ib) = 2a = \text{Re}z;$$

(b) Let $z = a + ib$, then

$$z - \overline{z} = (a + ib) - (a - ib) = 2ib = 2\text{Im}z;$$

c) Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, then

$$z_1\overline{z_2} = (a_1 + ib_1)(a_2 - ib_2) = (a_1a_2 + b_1b_2) + i(b_1a_2 - a_1b_2),$$

$$\overline{z_1}z_2 = (a_1 - ib_1)(a_2 + ib_2) = (a_1a_2 + b_1b_2) - i(b_1a_2 - a_1b_2)$$

and

$$z_1\overline{z_2} + \overline{z_1}z_2 = 2(a_1a_2 + b_1b_2).$$

Hence

$$2\text{Re}(z_1\overline{z_2}) = 2\text{Re}(\overline{z_1}z_2) = z_1\overline{z_2} + \overline{z_1}z_2 = 2(a_1a_2 + b_1b_2);$$

d) Let $z = a + ib$, then

$$|z| = \sqrt{a^2 + b^2} \geq \sqrt{a^2} = |a| \geq a = \text{Re}z;$$

e) Let $z = a + ib$, then

$$|z| = \sqrt{a^2 + b^2} \geq \sqrt{b^2} = |b| \geq b = \text{Im}z;$$

(f) Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, then using part (c) and part (d) we get,

$$|z_1\overline{z_2} + \overline{z_1}z_2| = 2|\text{Re}(z_1\overline{z_2})| \leq 2|z_1\overline{z_2}|;$$

g) Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, then

$$\left(|z_1| - |z_2|\right)^2 = \left(|z_1|^2 + |z_2|^2\right) - 2|z_1||z_2|;$$

$$|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1 + z_2}) = (z_1 + z_2)(\overline{z_1} + \overline{z_2})$$

$$= z_1\overline{z_1} + z_1\overline{z_2} + z_2\overline{z_1} + z_2\overline{z_2} = (|z_1|^2 + |z_2|^2) + (z_1\overline{z_2} + z_2\overline{z_1});$$

$$\left(|z_1| + |z_2|\right)^2 = \left(|z_1|^2 + |z_2|^2\right) + 2|z_1||z_2|. $$
To simplify let \( A = (|z_1|^2 + |z_2|^2) \). We want to show that \((|z_1| - |z_2|)^2 \leq |z_1 + z_2|^2 \leq (|z_1| + |z_2|)^2\). The above identities imply that this is equivalent to showing

\[
A - 2|z_1z_2| \leq A + (z_1\overline{z_2} + z_2\overline{z_1}) \leq A + 2|z_1z_2|.
\]

Hence we have to prove that

\[
-2|z_1z_2| \leq (z_1\overline{z_2} + z_2\overline{z_1}) \leq 2|z_1z_2|.
\]

Part (f) implies that

\[
|z_1\overline{z_2} + z_2\overline{z_1}| \leq 2|z_1z_2|,
\]

moreover it is always true that

\[
-|z_1\overline{z_2} + z_2\overline{z_1}| \leq (z_1\overline{z_2} + z_2\overline{z_1}) \leq |z_1\overline{z_2} + z_2\overline{z_1}|.
\]

Thus we conclude that

\[
-2|z_1z_2| \leq -|z_1\overline{z_2} + z_2\overline{z_1}| \leq (z_1\overline{z_2} + z_2\overline{z_1}) \leq |z_1\overline{z_2} + z_2\overline{z_1}| \leq 2|z_1z_2|
\]

which is what we wanted to show.

5. Express the following quantities in the form \(a + ib\), where \(a\) and \(b\) are real:
   (a) \((1 + i)^3\),
   (b) \(\frac{1+i}{1-i}\),
   (c) \(e^{\pi i/2}\),
   (d) \(e^{2+\pi i/4}\),
   (e) \(\sin(\frac{\pi}{4} + 2i)\),
   (f) \(\cosh(2 + \frac{\pi i}{4})\).

\textbf{Solution.} (a) \((1 + i)^3 = 1 + 3i + 3i^2 + i^3 = 1 + 3i - 3 - i = -2 + 2i;\)
   (b) \(\frac{1+i}{1-i} = \frac{1+i}{1-i} \cdot \frac{1+i}{1+i} = \frac{1+2i+i^2}{1-i+i-i} = \frac{2i}{2} = 0 + i;\)
   (c) \(e^{\pi i/2} = \cos \frac{\pi}{2} + i\sin \frac{\pi}{2} = 0 + i;\)
   (d) \(e^{2+\pi i/4} = e^2(e^{\pi i/4}) = e^2(\cos \frac{\pi}{4} + i\sin \frac{\pi}{4}) = e^2(\sqrt{2}/2 + i\sqrt{2}/2) = e^2\sqrt{2}/2 + ie^2\sqrt{2}/2;\)
   (e) \(\sin(\frac{\pi}{4} + 2i) = (\sin \frac{\pi}{4}\cos 2i) + i(\cos \frac{\pi}{4}\sin 2i) = \cos 2/\sqrt{2} + i\sinh 2/\sqrt{2};\)
   (f) \(\cosh(2 + \frac{\pi i}{4}) = (\cosh 2\cos \frac{\pi i}{4}) + i(\sinh 2\sin \frac{\pi i}{4}) = \cosh 2/\sqrt{2} + i\sinh 2/\sqrt{2}.\)

9. Prove that \(e^z\) possesses no zeros, that the zeros of \(\sin z\) and \(\cos z\) all lie on the real axis, and that those of \(\sinh z\) and \(\cosh z\) all lie on the imaginary axis.

\textbf{Solution.} We have to show that \(e^z \neq 0\) for all \(z\). Let \(z = x + iy\), then

\[
e^z = e^{x+iy} = e^x e^{iy} = e^x (\cos y + i \sin y).
\]

Since \(e^x \neq 0\) for all \(x\) real, \(e^z \neq 0\) if and only if \(\cos y = \sin y = 0\) for a real number \(y\), which is not possible. Hence \(e^z \neq 0\) for all \(z\).

Now we want to prove that the zeros of \(\sin z\) and \(\cos z\) are real. Let \(z = x + iy\), then (identity (32) pg 544)

\[
\sin(x + iy) = \sin x \cosh y + i(\cos x \sinh y).
\]

Therefore \(\sin(x + iy) = 0\) if and only if

\[
(1) \sin x \cosh y = 0
\]
and

$$(2) \cos x \sinh y = 0.$$  

Observe that

$$\cosh y = \frac{e^y + e^{-y}}{2} \neq 0,$$

for all $y$, therefore if (1) holds, we must have $\sin x = 0$ which implies $\cos x \neq 0$. Thus, if (2) holds, we must have

$$\sinh y = \frac{e^y - e^{-y}}{2} = 0$$

which implies $y = 0$. Hence the zeros of $\sin z$ are real.

Analogously, (identity (32) pg 544)

$$\cos(x + iy) = \cos x \cosh y + i(\sin x \sinh y).$$

Therefore if we assume $\cos(x + iy) = 0$, we must have

$$(1)' \cos x \cosh y = 0$$

and

$$(2)' \sin x \sinh y = 0.$$  

Since $\cosh y \neq 0$, for all $y$ if (1)' holds, we must have $\cos x = 0$ which implies $\sin x \neq 0$. Thus, if (2)' holds, we must have $\sinh y = 0$ which implies $y = 0$. Hence the zeros of $\cos z$ are real.

Finally we want to show that the zeros of $\sinh z$ and $\cosh z$ are imaginary. The following identity (identity (32) pg 544)

$$\sinh(x + iy) = \sinh x \cos y + i(\cosh x \sin y).$$

implies that $\sinh(x + iy) = 0$ if and only if

$$(3) \sinh x \cos y = 0$$

and

$$(4) \cosh x \sin y = 0.$$  

Since $\cosh x \neq 0$, for all $x$, (4) implies $\sin y = 0$ and so $\cos y \neq 0$. Hence, if (3) holds, we must have $\sinh x = 0$ which implies $x = 0$. Hence the zeros of $\sinh z$ are imaginary.

Analogously, (identity (32) pg 544)

$$\cosh(x + iy) = \cosh x \cosh y + i(\sinh x \sin y).$$

Therefore $\cosh(x + iy) = 0$ if and only if

$$(3)' \cosh x \cos y = 0$$

and

$$(4)' \sinh x \sin y = 0.$$  

Since $\cosh x \neq 0$, for all $x$, (3)' implies that $\cos y = 0$ and so $\sin y \neq 0$. Thus, if (4)' holds, we must have $\sinh x = 0$ which implies $x = 0$. Hence the zeros of $\cosh z$ are imaginary.
10.3. Other Elementary Functions.

12. Show that the $n$th roots of unity are of the form $\omega_n^k$ ($k = 0, 1, \ldots, n - 1$), where $\omega_n = \cos(2\pi/n) + i \sin(2\pi/n)$.

Solution. We need to solve the equation $z^n = 1$. The principal value of the argument $\theta$ of unity is $\theta_P = 0$. Hence, from the formula derived in class,

$$z = |1|^{1/n} e^{i(\theta_P + 2k\pi)/n} = \left(e^{i2\pi/n}\right)^k = (\omega_n)^k, \quad k = 0, 1, 2, \ldots, n - 1,$$

where $\omega_n = e^{i2\pi/n} = \cos(2\pi/n) + i \sin(2\pi/n)$.

13. Determine all possible values of the following quantities in the form $a + ib$, and in each case give also the principal value, assuming the definition (39):

(a) $\log(1 + i)$, (b) $(i)^{3/4}$, (c) $(1 + i)^{1/2}$.

Solution. (a) $\log(1 + i) = \log \sqrt{2} + i(2k\pi + \frac{\pi}{4})$, where $k$ is an integer. The principal value is $\log \sqrt{2} + i\frac{\pi}{4}$.

(b) $(i)^{3/4} = \sqrt[4]{(i)^3} = \sqrt[4]{e^{i(\frac{3\pi}{2}+2k\pi)}} = e^{i\left(\frac{3\pi}{8} + \frac{k\pi}{2}\right)} = \cos \left(\frac{3\pi}{8} + \frac{k\pi}{2}\right) + i \sin \left(\frac{3\pi}{8} + \frac{k\pi}{2}\right)$, where $k = 0, 1, 2, 3$. The principal value is $\cos \frac{3\pi}{8} + i \sin \frac{3\pi}{8}$.

(c) $(1 + i)^{1/2} = e^{\frac{1}{2}\log(1+i)} = e^{\frac{1}{2}(\log \sqrt{2} + i(2k\pi + \frac{\pi}{4}))) = \sqrt{2}(\cos \left(\frac{\pi}{8} + k\pi\right) + i \sin \left(\frac{\pi}{8} + k\pi\right))$, where $k = 0, 1$. The principal value is $\sqrt{2}(\cos \frac{\pi}{8} + i \sin \frac{\pi}{8})$. 