4. Series Solutions of Differential Equations: Special Functions

5. Obtain the general solution of each of the following differential equations in terms of Maclaurin series:
   (a) \( \frac{d^2y}{dx^2} = xy \),
   (b) \( \frac{d^2y}{dx^2} + x \frac{dy}{dx} - y = 0 \).

Solution. (a) Try the Maclaurin series \( y = \sum_{n=0}^{\infty} a_n x^n \) to get
\[
xy = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=0}^{\infty} a_{n-1} x^n, \quad a_{-1} = 0,
\]
\[
\frac{d^2y}{dx^2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n + 2)(n + 1) a_{n+2} x^n.
\]
The differential equation yields
\[
\sum_{n=0}^{\infty} [(n + 2)(n + 1) a_{n+2} - a_{n-1}] x^n = 0,
\]
which is satisfied by all \( x \) in some neighborhood of \( x_0 = 0 \). Hence, the recurrence formula (relation) for the coefficients \( a_n \) reads
\[
(n + 2)(n + 1) a_{n+2} = a_{n-1}; \quad a_{-1} = 0, \quad n = 0, 1, 2, 3, \ldots.
\]
Find the coefficients explicitly for various \( n \):
\[
n = 0: \quad a_2 = 0
\]
\[
n = 1: \quad 3 \cdot 2a_3 = a_0
\]
\[
n = 2: \quad 4 \cdot 3a_4 = a_1
\]
\[
n = 3: \quad 5 \cdot 4a_5 = a_2
\]
\[
n = 4: \quad 6 \cdot 5a_6 = a_3
\]
\[
n = 5: \quad 7 \cdot 6a_7 = a_4
\]
\[
n = 6: \quad 8 \cdot 7a_8 = a_5, \ldots.
\]
Notice that \( a_0 \) and \( a_1 \) are independent and arbitrary, while all coefficients \( a_2, a_5, a_8, \ldots a_{3n+2}, \ldots = 0 \).

Date: October 22, 2002.
The corresponding power series for \( y(x) \) reads as
\[
y(x) = a_1 \left( x + \frac{x^4}{4 \cdot 3} + \frac{x^7}{(3 \cdot 4) (6 \cdot 7)} + \cdots + \frac{x^{3n+1}}{(3 \cdot 4) (6 \cdot 7) \cdots [3n (3n+1)]} + \cdots \right)
\]
\[
+ a_0 \left( 1 + \frac{x^3}{2 \cdot 3} + \frac{x^6}{(2 \cdot 3) (5 \cdot 6)} + \cdots + \frac{x^{3n}}{(2 \cdot 3) (5 \cdot 6) \cdots [(3n+2) (3n+3)]} + \cdots \right).
\]

(b) Once again, we try the Maclaurin series \( y(x) \sum_{n=0}^{\infty} a_n x^n \) to get
\[
x \frac{dy}{dx} = \sum_{n=0}^{\infty} n a_n x^n, \quad \frac{d^2 y}{dx^2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n,
\]
which in turn lead to the equation
\[
\sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} + (n-1)a_n] x^n = 0,
\]
satisfied by all \( x \) in some neighborhood of \( x_0 = 0 \). It follows that
\[
(n+2)(n+1)a_{n+2} = -(n-1)a_n, \quad n = 0, 1, 2, 3, \ldots
\]
Write the ensuing coefficients explicitly:
\[
n = 0: \quad 2a_2 = a_0,
\]
\[
n = 1: \quad 3 \cdot 2a_3 = 0 \cdot a_1 = 0,
\]
\[
n = 2: \quad 4 \cdot 3a_4 = -a_2,
\]
\[
n = 3: \quad 5 \cdot 4a_5 = -2a_3 = 0,
\]
\[
n = 4: \quad 6 \cdot 5a_6 = -3a_4,
\]
\[
n = 5: \quad 7 \cdot 6a_7 = -4a_5 = 0.
\]
It follows that \( a_0 \) and \( a_1 \) are independent and arbitrary. Further, all coefficients with odd index are zero, with the exception of \( a_1 \) (since the right-hand side of the equation for \( n = 1 \) vanishes).

The final Maclaurin series for \( y(x) \) reads as
\[
y(x) = a_0 \left( 1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{1 \cdot 3 x^6}{6!} - \frac{1 \cdot 3 \cdot 5 x^8}{8!} + \cdots \right)
\]
\[
+ (-1)^n \frac{1 \cdot 3 \cdot \cdots (2n-1) x^{2n+2}}{(2n)!} + \cdots + a_1 x.
\]
Notice that the independent solution involving \( a_1 \) is \( u(x) = x \).

6. For each of the following equations, obtain the most general solution which is representable by a Maclaurin series:
(a) \( \frac{d^2 y}{dx^2} + y = 0, \)
(b) \( \frac{d^2 y}{dx^2} - (x - 3)y = 0, \)
(c) \( (1 - \frac{1}{2} x^2) \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - y = 0, \)
(d) \( x^2 \frac{d^2 y}{dx^2} - \frac{dy}{dx} + y = 0, \)
(e) \((x^2 + x)\frac{d^2 y}{dx^2} - (x^2 - 2)\frac{dy}{dx} - (x + 2)y = 0\).

Obtain three nonvanishing terms in each infinite series involved.

**Solution.** (a) With \(y(x) = \sum_{n=0}^{\infty} A_n x^n\), the recurrence formula for the coefficients \(A_n\) is

\[(n + 2)(n + 1)A_{n+2} + A_n = 0, \quad n = 0, 1, 2, 3, \ldots .\]

Specifically,

\[n = 0: \quad 2 \cdot 1 A_2 + A_0 = 0 \Rightarrow A_2 = -\frac{A_0}{2 \cdot 1},\]
\[n = 1: \quad 3 \cdot 2 A_3 + A_1 = 0 \Rightarrow A_3 = -\frac{A_2}{2 \cdot 3},\]
\[n = 2: \quad 4 \cdot 3 A_4 + A_2 = 0 \Rightarrow A_4 = -\frac{A_3}{3 \cdot 4} = \frac{A_0}{4!},\]
\[n = 3: \quad 5 \cdot 4 A_5 + A_3 = 0 \Rightarrow A_5 = -\frac{A_4}{5 \cdot 4} = \frac{A_1}{5!}, \ldots .\]

It follows that

\[y(x) = A_0 \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots \right) + A_1 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \right).\]

(b) Again, start with \(y(x) = \sum_{n=0}^{\infty} A_n x^n\) and \(xy(x) = \sum_{n=0}^{\infty} A_{n-1} x^n\), where \(A_{-1} = 0\), to arrive at the recurrence formula

\[(n + 2)(n + 1)A_{n+2} - A_{n-1} + 3A_n = 0; \quad A_{-1} = 0, \quad n = 0, 1, 2, \ldots .\]

Specifically,

\[n = 0: \quad 2 \cdot 1 A_2 + 3 A_0 = 0 \Rightarrow A_2 = -\frac{3}{1 \cdot 2} A_0,\]
\[n = 1: \quad 3 \cdot 2 A_3 - A_0 + 3 A_1 = 0 \Rightarrow A_3 = \frac{A_0}{2 \cdot 3} - \frac{A_1}{2},\]
\[n = 2: \quad 4 \cdot 3 A_4 - A_1 + 3 A_2 = 0 \Rightarrow A_4 = \frac{A_1}{3 \cdot 4} - \frac{A_2}{4} = \frac{A_1}{3 \cdot 4} + \frac{3 A_0}{8}, \ldots .\]

It follows that

\[y(x) = A_0 + A_1 x - \frac{3}{2} A_0 x^2 + \left(\frac{A_0}{6} - \frac{A_1}{2}\right) x^3 + \left(\frac{A_1}{12} + \frac{3 A_0}{8}\right) x^4 + \ldots \]

\[= A_0 \left(1 - \frac{3}{2} x^2 + \frac{1}{6} x^3 - \ldots \right) + A_1 \left(x - \frac{x^3}{2} + \frac{x^4}{12} - \ldots \right).\]

(c) With \(y(x) = \sum_{n=0}^{\infty} A_n x^n\), we get

\[x \frac{dy}{dx} = \sum_{n=0}^{\infty} n A_n x^n, \quad x^2 \frac{d^2 y}{dx^2} = \sum_{n=0}^{\infty} n(n - 1) A_n x^n,\]

and we find the recurrence formula

\[(n + 2)(n + 1)A_{n+2} - \frac{1}{2}(n - 1)(n - 2)A_n = 0.\]
Try different values of $n$:

\[
\begin{align*}
  n = 0 : & \quad 2 \cdot 1A_2 - A_0 = 0 \Rightarrow A_2 = \frac{A_0}{2}, \\
  n = 1 : & \quad 3 \cdot 2A_3 - 0 = 0 \Rightarrow A_3 = 0, \\
  n = 2 : & \quad 4 \cdot 3A_4 = 0, \\
  n = 3 : & \quad 5 \cdot 4A_5 = A_3 = 0, \\
  n = 4 : & \quad 6 \cdot 5A_6 - 3A_4 = 0 \Rightarrow A_6 = 0, \\
  n = 5 : & \quad 7 \cdot 6A_7 - 2 \cdot 3A_5 = 0 \Rightarrow A_7 = 0 \quad \text{etc.}
\end{align*}
\]

It follows that all coefficients $A_n$ with $n \geq 3$ vanish! Hence,

\[
y(x) = A_0 \left( 1 + \frac{x^2}{2} \right) + A_1 x.
\]

(d) Clearly,

\[
\begin{align*}
  \frac{dy}{dx} &= \sum_{n=0}^{\infty} (n+1)A_{n+1}x^n, \\
  x^2 \frac{d^2y}{dx^2} &= \sum_{n=0}^{\infty} n(n-1)A_n x^n.
\end{align*}
\]

The recurrence formula is

\[
[n(n-1)+1]A_n = (n+1)A_{n+1}, \quad n = 0, 1, 2, \ldots .
\]

Specifically,

\[
\begin{align*}
  n = 0 : & \quad A_0 = A_1, \\
  n = 1 : & \quad A_1 = 2A_2 \Rightarrow A_2 = \frac{A_0}{2}, \\
  n = 2 : & \quad 3A_2 = 3A_3 \Rightarrow A_3 = \frac{A_0}{2} \quad \text{etc.}
\end{align*}
\]

Hence,

\[
y(x) = A_0 \left( 1 + x + \frac{x^2}{2} + \ldots \right).
\]

(e) Clearly,

\[
\begin{align*}
  (x+2)y &= \sum_{n=0}^{\infty} A_{n-1}x^n + 2 \sum_{n=0}^{\infty} A_n x^n, \quad A_{-1} = 0, \\
  (x^2 - 2) \frac{dy}{dx} &= \sum_{n=0}^{\infty} (n-1)A_{n-1}x^n - 2 \sum_{n=0}^{\infty} (n+1)A_{n+1}x^n, \\
  (x^2 + x) \frac{d^2y}{dx^2} &= \sum_{n=0}^{\infty} n(n-1)A_n x^n + \sum_{n=0}^{\infty} n(n+1)A_{n+1}x^n.
\end{align*}
\]

By putting all these terms together, the recurrence formula reads

\[
(n - 2)(n + 1)A_n + (n + 1)(n + 2)A_{n+1} - nA_{n-1} = 0; \quad A_{-1} = 0, \quad n = 0, 1, 2, \ldots .
\]
Specifically,

\[ n = 0: \quad -2A_0 + 1 \cdot 2A_1 = 0 \Rightarrow A_0 = A_1, \]
\[ n = 1: \quad -2A_1 + 2 \cdot 3A_2 - A_0 = 0 \Rightarrow A_2 = \frac{A_0}{2} \quad \text{etc.} \]

Finally,

\[ y(x) = A_0 \left( 1 + x + \frac{x^2}{2} + \ldots \right). \]

4.3. Singular points of linear, second-order differential equations. .

8. Locate and classify the singular points of the following differential equations:
   (a) \((x - 1)y'' + \sqrt{x}y = 0\) \((x \geq 0),\)
   (b) \(y'' + y \log x + xy = 0\) \((x \geq 0),\)
   (c) \(xy'' + y \sin x = 0,\)
   (d) \(y'' - |1 - x^2|y = 0,\)
   (e) \(y'' + y \cos \sqrt{x} = 0\) \((x \geq 0).\)

Solution. (a) The singular points are \(x = 1\) and \(x = 0.\) \(x = 1\) is a regular singular point since \((x - 1)^2 \cdot \frac{\sqrt{x}}{(x-1)} = (x - 1)\sqrt{x}\) has a Taylor expansion near \(x = 1.\) Since \((x^2 \cdot \frac{\sqrt{x}}{(x-1)})'''|_{x=0}\) does not exist, \(x^2 \cdot \frac{\sqrt{x}}{(x-1)}\) does not have a Taylor expansion near \(x = 0.\) So \(x = 0\) is an irregular singular point.

   (b) The singular point is \(x = 0,\) which is irregular since \(x \log x\) is not differentiable at \(x = 0.\)

   (c) There are no singular points. (Note that \(\frac{\sin x}{x} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-2}}{(2n-1)!}.\))

   (d) The singular points are \(x = 1\) and \(x = -1.\) Since neither \((x - 1)^2 \cdot |1 - x^2|y''|_{x=1}\) nor \(((x + 1)^2 \cdot |1 - x^2|y''|_{x=-1}\) is well defined, both singular points are irregular.

   (e) There are no singular points. (Note that \(\cos \sqrt{x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(2n)!}.\))

4.4. The Method of Frobenius. .

11. Use the method of Frobenius to obtain the general solution of each of the following differential equations, valid near \(x = 0:\)
   (a) \(2xy'' + (1 - 2x)y' - y = 0,\)
   (b) \(x^2y'' + xy + (x^2 - \frac{1}{4})y = 0,\)
   (c) \(xy'' + 2y' + xy = 0,\)
   (d) \(x(1 - x)y'' - 2y' + 2y = 0.\)

Solution. (a) Rewrite the equation as

\[ y'' + \frac{1}{x} (\frac{1}{2} - x)y' + \frac{1}{x^2} (-\frac{x}{2})y = 0. \]
Then we can see that $P_0 = 1/2$, $P_1 = -1$, $Q_1 = -1/2$, and all other $P_n$'s, $Q_n$'s and $R_n$'s are zeros. So $f(s) = s^2 - \frac{1}{2}s$, $g_1(s) = -s + 1/2$, and $g_n(s) = 0$ if $n \neq 1$. $f(s) = 0$ has two roots: $s = \frac{1}{2}$ and $s = 0$. Take $s = 0$, then $A_n = \frac{A_{n-1}}{n}$, for all $n \geq 1$. Hence, by induction, $A_n = \frac{A_{n}}{n!}$ for all $n \geq 0$. Therefore

$$y = y_0 \sum_{n=1}^{\infty} \frac{x^n}{n!} = y_0 e^x$$

Now, take $s = 1/2$, then $A_n = 2\frac{A_{n-1}}{2n+1}$, for all $n \geq 1$. Therefore

$$y = x^{\frac{1}{2}} \sum_{n=0}^{\infty} A_n x^n = x^{\frac{1}{2}} A_0 \sum_{n=0}^{\infty} \frac{2^n}{(2n+1)!!} x^n.$$

Here $(2n+1)!! = 3 \cdot 5 \cdot 7 \cdots (2n+1)$.

The general solution is then of the form:

$$y(x) = C_1 e^x + C_2 x^{\frac{1}{2}} \left( \sum_{n=0}^{\infty} \frac{2^n}{(2n+1)!!} x^n \right).$$

(b) Rewrite the equation as

$$y'' + \frac{1}{x} y' + \frac{1}{x^2} (x^2 - \frac{1}{4}) y = 0.$$

Then we can see that $P_0 = 1$, $Q_0 = -\frac{1}{4}$, $Q_2 = 1$, and all other $P_n$'s, $Q_n$'s and $R_n$'s are zeros. So $f(s) = s^2 - \frac{1}{4}$, $g_2(s) = 1$, and $g_n(s) = 0$ if $n \neq 2$. $f(s) = 0$ has two roots: $s = \frac{1}{2}$ and $s = -\frac{1}{2}$.

For $s = -\frac{1}{2}$ we have $A_n = -\frac{1}{n(n-1)} A_{n-2}$ for all $n \geq 2$. From this, it easy to check by induction that $A_{2n} = (-1)^n A_0$ and $A_{2n+1} = (-1)^n (2n+1) A_1$ for all $n \geq 0$. So, in this case,

$$y = x^{-\frac{1}{2}} \sum_{n=0}^{\infty} A_n x^n = A_0 x^{-\frac{1}{2}} \left( \sum_{n=0}^{\infty} (-1)^n (2n)!! x^{2n} \right) + A_1 x^{-\frac{1}{2}} \left( \sum_{n=0}^{\infty} (-1)^n (2n+1)!! x^{2n+1} \right) = A_0 x^{-\frac{1}{2}} \cos x + A_1 x^{-\frac{1}{2}} \sin x.$$

The general solution is then of the form

$$y = c_0 x^{-\frac{1}{2}} \cos x + c_1 x^{-\frac{1}{2}} \sin x.$$

(c) Rewrite the equation as

$$y'' + \frac{2}{x} y' + \frac{x^2}{x^2} y = 0.$$

Then we can see that $P_0 = 2$, $Q_2 = 1$, and all other $P_n$'s, $Q_n$'s and $R_n$'s are zeros. So $f(s) = s^2 + s$, $g_2(s) = 1$, and $g_n(s) = 0$ if $n \neq 2$. $f(s) = 0$ has two roots: $s = -1$ and $s = 0$. 
For \( s = -1 \), we have \( A_n = -\frac{1}{n(n-1)} A_{n-2} \). So \( A_{2n} = \frac{(-1)^n}{(2n)!} A_0 \) and \( A_{2n+1} = \frac{(-1)^n}{(2n+1)!} A_1 \) for all \( n \geq 0 \). Then

\[
y = x^{-1} \sum_{n=0}^{\infty} A_n x^n = x^{-1}(A_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} + A_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n})
\]

\[
= x^{-1}(A_1 \sin x + A_0 \cos x).
\]

The general solution is then of the form

\[
y = x^{-1}(c_1 \sin x + c_0 \cos x).
\]

(d) Rewrite the equation as

\[
(1-x)y'' - \frac{2}{x}y' + \frac{2x}{x^2} y = 0.
\]

Then we can see \( R_1 = -1, P_0 = -2, Q_1 = 2 \), and all other \( P_n \)'s, \( Q_n \)'s and \( R_n \)'s are zeros. So \( f(s) = s^2 - 3s, g_1(s) = -s^2 + 3s, \) and \( g_n(s) = 0 \) for all \( n > 1 \). \( f(s) \) has two roots: \( s = 3 \) and \( s = 0 \).

For \( s = 0 \), \( A_n = -\frac{g_1(n)}{f(n)} A_{n-1} = A_{n-1} \) for all \( n \geq 1, n \neq 3 \). Thus, \( A_2 = A_1 = A_0 \), and \( A_3 = A_4 = A_5 = \cdots \). So, in this case,

\[
y = x^0 \sum_{n=0}^{\infty} A_n x^n
\]

\[
= A_0 (1 + x + x^2) + A_3 x^3 \sum_{n=0}^{\infty} x^n
\]

\[
= A_0 \frac{1 - x^3}{1 - x} + A_3 x^3 \frac{1}{1 - x}.
\]

The general solution is then of the form

\[
y = c_0 \frac{1}{1 - x} + c_1 \frac{x^3}{1 - x}.
\]

12. Use the method of Frobenius to obtain the general solution of each of the following differential equations, valid near \( x = 0 \):

(a) \( x^2 y'' - 2xy' + (2 - x^2) y = 0 \),
(b) \( (x - 1)y'' - xy' + y = 0 \),
(c) \( xy'' - y' + 4x^3 y = 0 \),
(d) \( (1 - \cos x)y'' - \sin xy' + y = 0 \).

Solution. (a) Rewrite the equation as

\[
y'' - \frac{2}{x}y' + \frac{1}{x^2}(2 - x^2) y = 0.
\]
Then we can see that \( P_0 = -2, Q_0 = 2, Q_2 = -1 \) and all other \( P_n \)'s, \( Q_n \)'s and \( R_n \)'s are zeros. So \( f(s) = s^2 - 3s + 2, g_2(s) = -1, \) and \( g_n(s) = 0 \) if \( n \neq 2. f(s) = 0 \) has two roots: \( s = 1 \) and \( s = 2. \) For \( s = 1, \) we have

\[
A_n = \frac{A_{n-2}}{n(n-1)}
\]

for \( n \geq 2. \) From this, it's easy to check by induction that \( A_{2n} = \frac{A_n}{(2n)!} \) and \( A_{2n+1} = \frac{A_1}{(2n+1)!} \)

for all \( n \geq 0. \) So

\[
y = x \left( A_0 \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n} + A_1 \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1} \right) = x(A_0 \cosh(x) + A_1 \sinh(x)).
\]

The general solution is then of the form

\[
y = c_0 x \cosh(x) + c_1 x \sinh(x).
\]

(b) Rewrite the equation as

\[
(1-x)y'' + xy' - \frac{x^2}{x^2} y = 0.
\]

Then we can see that \( R_1 = -1, P_2 = 1, Q_2 = -1 \), and all other \( P_n \)'s, \( Q_n \)'s and \( R_n \)'s are zeros. So \( f(s) = s^2 - s, g_1(s) = -(s-1)(s-2), g_2(s) = s-3, \) and \( g_n(s) = 0 \) if \( n \geq 3. f(s) = 0 \) has two roots: \( s = 0 \) and \( s = 1. \)

For \( s = 0, \) we have

\[
A_n = -\frac{g_1(n) A_{n-1} + g_2(n) A_{n-2}}{f(n)} = \frac{n-2}{n} A_{n-1} - \frac{n-3}{n(n-1)} A_{n-2}
\]

for \( n \geq 2. \) From this, it's easy to check by induction that \( A_n = \frac{A_0}{n!} \) if \( n \geq 2. \) So

\[
y = A_0 \left( 1 + \sum_{n=2}^{\infty} \frac{n^n}{n!} \right) + A_1 x = A_0 (e^x - x) + A_1 x = A_0 e^x + (A_1 - A_0)x.
\]

Hence the general solution is of the form

\[
y = c_0 e^x + c_1 x.
\]

(c) Rewrite the equation as

\[
y'' - \frac{1}{x} y' + \frac{4x^4}{x^2} y = 0.
\]

Then we can see that \( Q_4 = 4, P_0 = -1, \) and all other \( P_n \)'s, \( Q_n \)'s and \( R_n \)'s are zeros. So \( f(s) = s^2 - 2s, g_4(s) = 4, \) and \( g_n(s) = 0 \) if \( n \neq 4. f(s) = 0 \) has two roots: \( s = 0 \) and \( s = 2. \)

For \( s = 0, \) we have \( A_1 = A_3 = 0, \) and \( A_n = -\frac{(-1)^n}{(2n)!} A_{n-4} \) for all \( n \geq 4. \) From these, it's easy to check by induction that \( A_{2n+1} = 0, A_{4n} = \frac{(-1)^n}{(2n)!} A_0, \) and \( A_{4n+2} = \frac{(-1)^n}{(2n+1)!} A_2 \) for all \( n \geq 0. \) So

\[
y = A_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{4n} + A_2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{4n+2} = A_0 \cos(x^2) + A_2 \sin(x^2).
\]
The general solution is then of the form

\[ y = c_0 \cos(x^2) + c_1 \sin(x^2). \]

(d) Rewrite the equation as

\[
\left( \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 2)!} x^{2n} \right) y'' + \frac{1}{x} \left( \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n + 1)!} x^{2n} \right) y' + \frac{2}{x^2} y = 0.
\]

Then we can see that \( Q_0 = 2, P_{2n} = 2 \frac{(-1)^{n+1}}{(2n + 1)!}, \) \( R_{2n} = 2 \frac{(-1)^n}{(2n + 2)!} \) for all \( n \geq 0, \) and all other \( P_n \)'s, \( Q_n \)'s and \( R_n \)'s are zeros. So \( f(s) = (s - 1)(s - 2), \) and \( g_{2n-1}(s) = 0, g_{2n}(s) = 2 \frac{(-1)^n}{(2n + 2)!} (s - 2n)(s - 4n - 3) \) for all \( n \geq 1. \) \( f(s) = 0 \) has two roots: \( s = 1 \) and \( s = 2. \)

For \( s = 1, \) using the equation

\[ f(s + n)A_n = - \sum_{k=1}^{n} g_k(s + n)A_{n-k}, \]

it's easy to check by induction that \( A_{2n} = \frac{(-1)^n}{(2n + 1)!} A_0, \) and \( A_{2n+1} = 2 \frac{(-1)^n}{(2n + 2)!} A_1 \) for all \( n \geq 0. \) So

\[
y = x \sum_{n=0}^{\infty} A_n x^n
\]

\[ = A_0 x \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)!} x^{2n} + A_1 x \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 2)!} x^{2n+1}
\]

\[ = A_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)!} x^{2n+1} + A_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 2)!} x^{2n+2}
\]

\[ = A_0 \sin x + 2A_1 (1 - \cos x).
\]

The general solution is then of the form

\[ y = c_0 \sin x + c_1 (1 - \cos x). \]