NAME: SOLUTIONS

18.075 In-class Exam #1
Wednesday, September 29, 2004

Justify your answers. Cross out what is not meant to be part of your final answer. Total number of points: 45.

I. (5 pts) Show that for any complex numbers $z_1$ and $z_2$,

$$||z_1| - |z_2|| \leq |z_1 + z_2|.$$ 

It suffices to show that

$$|z_1 - z_2|^2 \leq |z_1 + z_2|^2 = (\bar{z}_1 + \bar{z}_2) \cdot (z_1 + z_2) = |\bar{z}_1|^2 + |z_2|^2 + 2\text{Re}(\bar{z}_1 z_2)$$

The LHS equals

$$|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1 z_2| = |z_1|^2 + |z_2|^2 - 2|\bar{z}_1 z_2|.$$

Hence, we have to show that

$$|z_1|^2 + |z_2|^2 - 2|\bar{z}_1 z_2| \leq |\bar{z}_1|^2 + |z_2|^2 + 2\text{Re}(\bar{z}_1 z_2)$$

$$\Leftrightarrow |\bar{z}_1 z_2| \geq -\text{Re}(\bar{z}_1 z_2).$$

Let $w = \bar{z}_1 z_2 : |w| \geq -\text{Re}w$.

If $w = u + iv$, $u,v:\text{real}$, then we have to show that $\sqrt{u^2 + v^2} \geq -u$

For $u > 0$, this statement is obviously true.

For $u \leq 0$, $0 \leq -u \leq \sqrt{u^2 + v^2} \iff u^2 \leq u^2 + v^2 \iff v^2 > 0 :$ true.

Hence, we proved that $||z_1| - |z_2|| \leq |z_1 + z_2|$. 

II. (5 pts) Find all possible values of 

\((-\sqrt{3} + i)^{1/5}\).

Let \(z = -\sqrt{3} + i = r e^{i \theta_p}; \quad -\pi < \theta_p \leq \pi\).

\(r = |z| = \sqrt{3 + 1} = 2\)

\(\theta_p = \arctan \left( \frac{1}{-\sqrt{3}} \right) = \left\{ \begin{array}{l}
-\pi/6 \\
\pi - \pi/6 = \frac{5\pi}{6}
\end{array} \right., \quad \text{of which we}
\)

\(\theta_p = \frac{5\pi}{6}\) because \(z\) lies in the 2nd quadrant.

\(z^{1/5} = \left( 2 e^{i \frac{5\pi}{6} + i2kn} \right)^{1/5} = \sqrt[5]{2} \cdot e^{i \frac{\pi}{6} + i \frac{2k}{5} \pi}, \quad > 0\)

where \(k = 0, 1, 2, 3, 4\).
III.

1. (3 pts) Can the function \( u(x, y) = x^2 - y^2 - x - y \) be the REAL part of an analytic function \( f(z) = u(x, y) + iv(x, y) \)? **Hint:** You may use the Laplace equation, if you wish.

We check whether \( u(x,y) \) satisfies the Laplace eqn.

\[
\frac{\partial^2 u}{\partial x^2} = 2, \quad \frac{\partial^2 u}{\partial y^2} = -2
\]

\[
\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.
\]

So, yes, it is possible that \( u \) is the real part of an analytic function.

2. (5 pts) Determine all functions \( v(x, y) \) such that \( f(z) = u(x, y) + iv(x, y) \) is analytic.

We apply the Cauchy-Riemann eqs: \[ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \]

(1): \[ \frac{\partial v}{\partial y} = 2x - 1 \Rightarrow v(x,y) = 2xy - y + C(x) \]

(2): \[ -2y - 1 = -2y - C'(x) \Rightarrow C'(x) = 1 \Rightarrow C(x) = x + k \quad \text{real const.} \]

So, \[ v(x,y) = 2xy - y + x + k \]
3. (3 pts) Find explicitly as a function of \( z \) the \( f(z) \) such that

\[
f(z) = u(x, y) + iv(x, y).
\]

\[
f(z) = x^2 - y^2 - x - y + i(2xy - y + x + k)
\]

\[
= (x^2 - y^2 + i2xy) - (x + iy) + i(x + iy) + ik
\]

\[
= (x + iy)^2 + (-1+i)(x + iy) + ik
\]

\[
= z^2 + (-1+i)z + ik, \quad k: \text{real const.}
\]
IV. (6 pts) Compute the line integral

\[ \int_C \frac{z^3 + z^2 + z + 1}{z^4} \, dz \]

where \( C \) is the LOWER half-circle centered at 0 joining \( -\frac{1-i}{\sqrt{2}} \) and \( \frac{1-i}{\sqrt{2}} \) in the positive (counterclockwise) sense.

\[ I = \int_C \frac{z^3 + z^2 + z + 1}{z^4} \, dz = \int_C \frac{dz}{z} + \int_C \frac{dz}{z^2} + \int_C \frac{dz}{z^3} + \int_C \frac{dz}{z^4} \]

\[ = \ln(z) \bigg|_{-\frac{1-i}{\sqrt{2}}}^{\frac{1-i}{\sqrt{2}}} = -\frac{\pi}{4} - \left(-i \frac{3\pi}{4}\right) = i \frac{\pi}{2} \]

\[ \int_C \frac{dz}{z^2} = -\frac{1}{z^2} \bigg|_{-\frac{1-i}{\sqrt{2}}}^{\frac{1-i}{\sqrt{2}}} = -e^{-\frac{i\pi}{4}} + e^{i\frac{3\pi}{4}} = -\frac{1+i}{\sqrt{2}} + \frac{1+i}{\sqrt{2}} = -\sqrt{2} \]

\[ \int_C \frac{dz}{z^3} = -\frac{1}{2} \frac{1}{z^3} \bigg|_{-\frac{1-i}{\sqrt{2}}}^{\frac{1-i}{\sqrt{2}}} = -\frac{1}{2} \left( e^{i\frac{\pi}{2}} - e^{i\frac{3\pi}{2}} \right) = -\frac{1}{2} (i+i) = -i \]

\[ \int_C \frac{dz}{z^4} = -\frac{1}{3} \frac{1}{z^4} \bigg|_{-\frac{1-i}{\sqrt{2}}}^{\frac{1-i}{\sqrt{2}}} = -\frac{1}{3} \left( e^{i\frac{3\pi}{4}} - e^{i\frac{9\pi}{4}} \right) = -\frac{1}{3} e^{i\frac{3\pi}{4}} \left( e^{-i\frac{3\pi}{4}} - e^{-i\frac{3\pi}{4}} \right) \]

\[ = \frac{i}{3} \left(-2i\right) \sin \frac{3\pi}{4} = \frac{2}{3} \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{3} \]

So:

\[ I = i \frac{\pi}{2} - \sqrt{2} - i + \frac{\sqrt{2}}{3} = -\frac{2\sqrt{2}}{3} + i \left( \frac{\pi}{2} - 1 \right) \]
V. Let

\[ f(z) = \frac{1}{(2-z)(z+3)}. \]

1. (2 pts) Write \( f(z) \) as a sum of fractions, i.e.,

\[ f(z) = \frac{A}{z-2} + \frac{B}{z+3}; \]

\[ A = \lim_{z \to 2} \left( (z-2) f(z) \right) = -\lim_{z \to 2} \frac{1}{z+3} = -\frac{1}{5} \]

\[ B = \lim_{z \to -3} \left( (z+3) f(z) \right) = \lim_{z \to -3} \frac{1}{z-2} = \frac{1}{5} \]

\[ f(z) = -\frac{1}{5} \frac{1}{z-2} + \frac{1}{5} \frac{1}{z+3} \]

\( f(z) \) has singular points at \( z = 2, -3 \).

2. (3 pts) Explain whether it is possible to expand \( f(z) \) in Laurent (or Taylor) power series of:

(i) \( z \), that converges in \( 0 \leq |z| < 3 \)?

\[ f(z) \text{ "blows up" at } z = 2, -3: \text{ it is NOT analytic there} \]

The region \( 0 \leq |z| < 3 \) encloses \( z = 2 \).

So, we can NOT expand \( f(z) \) in Laurent series in this region.

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(ii) $z$, that converges in $3 < |z|$?

$f(z)$ is free of singular points in this region; so, $f(z)$ is analytic for $|z| > 3$ and we can expand it in Laurent series there.

(iii) $z + 1$, that converges in $1 < |z + 1| < 4$?

The region $1 < |z + 1| < 4$ encloses the points $z = 2, -3$.

So, $f(z)$ is not analytic in $1 < |z + 1| < 4$,

and therefore cannot be expanded in Laurent series.
3. (4 pts) Write the Laurent series expansion of \( f(z) \) for \( 5 < |z - 2| < \infty \) as a power series of \( (z - 2) \).

\[
\frac{1}{(2-z)(z+3)}
\]

Let \( w = z - 2 \) : \( f(z) = \frac{1}{-w(w+5)} \), where \( |w| > 5 \).

\[
f(z) = -\frac{1}{w^2} \left( \frac{1}{1 + \frac{5}{w}} \right) = -\frac{1}{w^2} \left[ 1 - \frac{5}{w} + \left( \frac{5}{w} \right)^2 + \cdots + (-1)^n \left( \frac{5}{w} \right)^n + \cdots \right]
\]

\[\lambda : |\lambda| < 1 \]

\[
= -\frac{1}{w^2} \sum_{n=0}^{\infty} (-1)^n \left( \frac{5}{w} \right)^n = -\sum_{n=0}^{\infty} (-1)^n \frac{5^n}{(z-2)^{n+2}}
\]

\[
\therefore f(z) = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{5^n}{(z-2)^{n+2}}
\]
VI. (6 pts) Let
\[ f(z) = \frac{1}{(z^2 + z)(z + 2)^3}. \]
Compute the integral of \( f(z) \) on the circles of center 1 and radii 1/2, 3/2, and 100, respectively.

\( f(z) \) has singular points at \( z = 0, -1, -2 \).

\[ \text{Radii:} \quad R = \frac{1}{2} \; ; \quad \text{the circle} \; C_1 \; \text{encloses none of the singular points.} \]

So, for \( R = \frac{1}{2} \), \[ \oint_C dz \, f(z) = 0, \quad \text{by the Cauchy integral theorem.} \]

- \( R = \frac{3}{2} \); the circle \( C_2 \) encloses the singular point \( z = 0 \) only.

\[ \oint_C dz \, f(z) = \oint_C dz \, \frac{1}{(z+1)(z+2)^3} = 2\pi i \cdot q(0), \quad \text{by the Cauchy integral formula.} \]

\[ \Rightarrow \oint_C dz \, f(z) = 2\pi i \cdot \frac{1}{(0+1)(0+2)^3} = \frac{\pi i}{4}. \]

- \( R = 100 \); the circle \( C_3 \) encloses all singular points. Because \( f(z) \) is analytic for \(|z| > R\), \( C_3 \) can be deformed with \( R \to \infty \), without change in the result of integration.

\[ \oint_C dz \, f(z) = \oint_{C(100)} \frac{dz}{z^2 + z^3} = 0, \quad \text{because} \quad z^2 + z^3 = z \quad \text{and} \quad \oint_C z^n \, dz = 0 \; \text{for} \; n \neq -1 \quad \text{where} \quad C \quad \text{encloses} \; 0. \]
VII. (3 pts) Determine where in the complex plane the following functions are analytic ($\bar{z}$ is the complex conjugate of $z$):

(i) \( \frac{e^z}{\sin z} \)

This function is analytic at all $z$ where $\sin z \neq 0$.

\[ \Rightarrow z \neq n\pi, \quad n: \text{integer}. \]

(ii) $z(\bar{z} + i)$

This function depends explicitly on $\mathbb{E}$, which is NOT analytic anywhere. So, the function is NOT analytic anywhere.

(iii) $e^{\frac{1}{z-1}}$

Let $w = \frac{1}{z-1}$

\[ e^{\frac{1}{z-1}} = e^w = 1 + w + \ldots + \frac{w^n}{n!} + \ldots : \text{converges} \]

for all $w \neq 0 \iff z \neq 1$.

So, the function is analytic for $z \neq 1$. 

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VIII. (3 pts-BONUS) Determine the constant \( A \) so that the following function is analytic everywhere.

\[
f(z) = \begin{cases} 
  A \frac{\cosh z - 1}{z^2} & \text{if } z \neq 0 \\
  1 & \text{if } z = 0.
\end{cases}
\]

For \( z \neq 0 \), \( f(z) = A \frac{\cosh z - 1}{z^2} \); so \( f(z) \) is a ratio of two analytic functions and it is also analytic itself.

For \( z \to 0 \), \( f(z) = A \frac{(1 + \frac{z^2}{2} + \ldots) - 1}{z^2} = \frac{A}{2} \).

So, we must have \( \frac{A}{2} = f(0) = 1 \iff A = 2 \).

For this value of \( A \), \( f(z) \) has a limit as \( z \to 0 \) that agrees with its value at \( z = 0 \). So, \( f(z) \) has a Taylor series at \( z = 0 \), and this series converges for all \( z \) (as it would for \( \cosh z \)).

It follows that, for \( A = 2 \), \( f(z) \) is analytic everywhere.