(1) It is sufficient to show that

$$|z_1 + z_2|^2 = (|z_1| + |z_2|)^2 = |z_1|^2 + |z_2|^2 + 2 \text{Re}(z_1 \overline{z_2})$$

where

$$|z_1 + z_2|^2 = (z_1 + \overline{z_2}) \cdot (\overline{z_1} + z_2) = |z_1|^2 + |z_2|^2 + 2 \text{Re}(z_1 \overline{z_2})$$

$$\Leftrightarrow \text{Re}(z_1 \overline{z_2}) \leq |z_1| |z_2|$$.

Let $w = z_1 \overline{z_2} = u + iv$. The last inequality is equivalent to

$$u \leq \sqrt{u^2 + v^2}$$.

This is true for all $u$: if $u < 0$, it is obviously true.

If $u > 0$, the last condition is equivalent (by squaring) to

$$u^2 \leq u^2 + v^2 \Leftrightarrow v^2 > 0$$.

(2) Let $z = 1 - \sqrt{3} i$. We need to find $z^{\frac{1}{3}}$.

We find $\theta_p$ for $z$.

$$\tan \theta_p = -\sqrt{3}$$, and $z$ lies in the 4th quadrant.

$$\Rightarrow \theta_p = -\frac{\pi}{3}$$ if we take $-\pi < \theta_p \leq \pi$.

The magnitude of $z$ is $|z| = \sqrt{1 + 3} = 2$.

Thus,

$$z^{\frac{1}{3}} = (2 \cdot e^{i \pi + 2k\pi})^{\frac{1}{3}} = \sqrt[3]{2^{\frac{1}{3}}} \cdot e^{i \frac{\theta_p}{3} + i \frac{2k\pi}{3}}, \quad k=0,1,2$$

$$= 2^{\frac{1}{9}} \cdot e^{i \frac{\pi}{3}} \cdot e^{i \frac{2k\pi}{3}}, \quad k=0,1,2$$

$$= -1$$.
\( u = 4xy + y \)

We check whether \( u \) can satisfy the Laplace equation, \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \)

\[ \frac{\partial^2 u}{\partial x^2} = 0, \quad \frac{\partial^2 u}{\partial y^2} = 0 \implies \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \] so \( u \) can be the imaginary part of an analytic function.

2. Suppose \( u + iv \) is analytic. Then \( u \) and \( v \) satisfy the Cauchy–Riemann equations:

\[ 1 \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad 2 \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \]

\( 1 \Rightarrow \frac{\partial u}{\partial x} = 4x+1 \implies u(x,y) = 4 \frac{x^2}{2} + x + c(y) = 2x^2 + x + c(y) \) \( 1' \)

\( 2 \Rightarrow \frac{\partial u}{\partial y} = -4y \equiv c'(y) = -4y \Rightarrow c(y) = -2y^2 + K. \)

So, \( u(x,y) = 2x^2 + x - 2y^2 + K, \) \( K = \text{real const.} \)

3. \( f(z) = u + iv = 2x^2 + x - 2y^2 + K + i(4xy + y) \)

\[ = (2x^2 - 2y^2 + i4xy) + x + iy + K \]

\[ = 2(x^2 + y^2) + x + iy + K = 2z^2 + z + K, \quad K: \text{real const.} \]
\[ I = \int_C \frac{z^3 - 2}{z^4} \, dz \]

Since it is not specified in the problem statement, we take \( C \) to be described in the counterclockwise (positive) sense.

Along \( C \), \( z = 2e^{i\theta} \), \( \frac{\pi}{2} < \theta < \frac{3\pi}{2} \)

\[ I = \int_C \frac{dz}{z^2} - 2 \int_C \frac{dz}{z^4} = \int_{\pi/2}^{3\pi/2} \frac{2ie^{i\theta} \, d\theta}{2e^{i\theta}} - 2 \int_{\pi/2}^{3\pi/2} \frac{2ie^{i\theta} \, d\theta}{(2e^{i\theta})^4} \]

\[ = i \cdot \left( \frac{3\pi}{2} - \frac{\pi}{2} \right) - 2 \cdot \frac{1}{2^3} \int_{\pi/2}^{3\pi/2} \, e^{-3i\theta} \]

\[ = i \cdot \pi - \frac{i}{2^2} \left( \frac{1}{3i} \right) \cdot e^{-3i\theta} \]

\[ = i \cdot \pi - \frac{i}{2^3} \left( e^{3i\pi/2} - e^{-3i\pi/2} \right) \]

So,
\[ I = i\pi + \frac{1}{3 \cdot 2^2} \left( e^{3i\pi/2} - e^{-3i\pi/2} \right) \]

\[ = i\pi + \frac{1}{3 \cdot 2^2} \left( e^{i\pi/2} - e^{-i\pi/2} \right) = i\pi + \frac{1}{3 \cdot 2^2} (-2i) = i\pi - \frac{i}{6} = i \left( \pi - \frac{1}{6} \right) \]

Alternative method: Notice that \( \frac{1}{z} = \frac{d}{dz} \ln z \), \( \frac{1}{z^4} = -\frac{d}{dz} \frac{1}{2^3} \).

So:
\[ I = \ln z \bigg|_{z=2i}^{z=-2i} + \frac{2}{3} \cdot \frac{1}{2^3} \bigg|_{z=-2i}^{z=2i} \]

\[ = \ln(-2i) - \ln(2i) + \frac{2}{3} \left[ \frac{1}{(2i)^3} - \frac{1}{(-2i)^3} \right] = i \left( \frac{3\pi}{2} - \frac{\pi}{2} \right) + \frac{2}{3} \cdot \frac{1}{2^3} = i \left( \pi - \frac{1}{6} \right) \]
\( f(z) = \frac{1}{z^2 - 5z + 6} = \frac{1}{(z-2)(z-3)} = \frac{A}{z-2} + \frac{B}{z-3} \)

\( z_1 = 2, \ z_2 = 3 \)

**A:** Multiply by \( z-2 \) and take \( z \to 2 \):

\[ A = \lim_{z \to 2} \left[ f(z)(z-2) \right] = \lim_{z \to 2} \frac{1}{z-3} = -1 \]

**B:** Multiply by \( z-3 \) and let \( z \to 3 \):

\[ B = \lim_{z \to 3} \left[ f(z)(z-3) \right] = \frac{1}{3-2} = 1. \]

So,

\[ f(z) = \frac{-1}{z-2} + \frac{1}{z-3} \]

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2. The region \( 0 \leq |z| < 3 \) contains the singular point \( z = 2 \),

so we cannot expand \( f(z) \) in Laurent series in this region.

(iii) The region \( 2 < |z+1| < 3 \) does not contain any singular points of \( f(z) \), so \( f(z) \) is analytic in this region. So, we can expand \( f(z) \) in Laurent series in this region.

(iii) The region \( |z+1| > 3 \) contains the singular point \( z = 3 \),

so we cannot expand \( f(z) \) in Laurent (or Taylor) series in this region.
3. Let \( w = z - 2 \Rightarrow z = 2 + w \), where \( 0 < |w| < 1 \)

\[
f(z) = \frac{1}{w(w-1)} = \frac{-1}{w} \cdot \frac{1}{1-w} = -\frac{1}{w} \sum_{n=0}^{\infty} w^n = -\frac{1}{z-2} \sum_{n=0}^{\infty} (z-2)^n
\]

\[\Rightarrow f(z) = -\sum_{n=0}^{\infty} (z-2)^{n-1} = -\frac{1}{z-2} \sum_{n=1}^{\infty} (z-2)^{n-1}
\]

This is a Laurent series for \( f(z) \), convergent for \( 0 < |z-2| < 1 \).

\[\text{(VI)} \quad f(z) = \frac{1}{(z^2+2)(z^2+3)}
\]

The singular ("bad") points of this function, where it ceases to be analytic, occur at \( z^2+2 = 0 \Rightarrow z = \pm \sqrt{2} i \), \( z^2+3 = 0 \Rightarrow z = \pm \sqrt{3} i \).

\[\text{(A)} \quad \text{The circle} \ C \text{ with center} -i \text{ and radius} \ r = \frac{1}{2} \text{ does NOT contain any singular point of} \ f(z). \text{ By the Cauchy integral theorem,}
\]

\[
\oint_C dz \ f(z) = 0.
\]

\[\text{(B)} \quad \text{With} \ r = 1, \text{ the circle} \ C \text{ contains the singular points} -i\sqrt{2}, -i\sqrt{3}.
\]

Clearly,

\[
\oint_C dz \ f(z) = \oint_{C_1} dz \ f(z) + \oint_{C_2} dz \ f(z)
\]

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where $C_1$ is a small circle centered at $-i\sqrt{2}$, and $C_2$ is a small circle centered at $-i\sqrt{3}$. We used the fact that $f(z)$ is analytic in the region between $C_1$, $C_1$ and $C_2$.

We apply the Cauchy integral formula to calculate $\oint_{C_1} f(z)$ and $\oint_{C_2} f(z)$:

\[ I_1 = \oint_{C_1} \frac{f(z)}{z - z_1} \, dz, \quad z_1 = -i\sqrt{2} \]

\[ f_1(z) = \frac{1}{(z - i\sqrt{2})(z^2 + 3)} \]

\[ = 2\pi i \cdot f_1(z_1) = 2\pi i \cdot \frac{1}{-2i\sqrt{2}(3 - 2)} = \frac{\pi}{\sqrt{2}} \]

\[ I_2 = \oint_{C_2} \frac{f(z)}{z - z_2} \, dz, \quad z_2 = -i\sqrt{3} \]

\[ f_2(z) = \frac{1}{(z - i\sqrt{3})(z^2 + 2)} \]

\[ = 2\pi i \cdot f_2(z_2) = 2\pi i \cdot \frac{1}{+2i\sqrt{3}(4i)} = \frac{\pi}{\sqrt{3}} \]

\[ \oint_{C} f(z) \, dz = I_1 + I_2 = \pi \left( \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{2}} \right) \]

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© With $p = 4$, the circle $C$ contains all singular points of $f(z)$ in its interior. So, $f(z)$ is analytic everywhere outside $C$. Thus, we can modify $C$ taking its radius $\to \infty$ without changing the result of integration. For large $|z|$ in the integrand

$$\oint_{C} dz \, f(z) \rightarrow \oint_{C} dz \, \frac{1}{(z^2 + 1)(z^2 + 2)} = \oint_{\text{large circle}} \frac{dz}{z^4} = 0,$$

because it's an integral $\oint dz \cdot z^n$ with $n \neq -1$, where $C$ contains $0$.

### V

1. $f(z) = e^{z^2} \sin z$.

$f(z)$ is analytic everywhere. By Cauchy integral theorem,

$$\oint_{C} dz \, f(z) = 0.$$

2. $f(z) = \frac{1}{z^{10}}$. Let $z = e^{i\theta}$, $0 \leq \theta < 2\pi$

$$\oint_{C} dz \, f(z) = \int_{0}^{2\pi} \frac{e^{i\theta} d\theta}{e^{10i\theta}} = \int_{0}^{2\pi} e^{-i9\theta} d\theta = \left. -\frac{i}{9} e^{-i9\theta} \right|_{\theta=0}^{2\pi} = 0.$$

3. $f(z) = \tan z = \frac{\sin z}{\cos z}$; "bad points" at $\cos z = 0$ $\Rightarrow z = (2\pi + 1)\frac{n}{2}$, $n$: integer

The circle with does NOT contain any of these points $\Rightarrow \oint_{C} dz \, f(z) = 0$

by Cauchy integral theorem.
See Lecture Notes

Basic Steps: Modify contour $C$ to a circle of radius $\varepsilon$ around $z = b$. Let $z - b = \varepsilon e^{i\theta}$, $0 \leq \theta < 2\pi$.

\[
\oint \frac{f(z)}{z - b} \, dz = \int_0^{2\pi} \frac{f(b + \varepsilon e^{i\theta})}{\varepsilon e^{i\theta}} \varepsilon i e^{i\theta} \, d\theta = i \int_0^{2\pi} f(b + \varepsilon e^{i\theta}) \, d\theta
\]

$\varepsilon \to 0 \quad i \cdot 2\pi \cdot f(b)$

(Explain why it is legitimate to modify $C$ this way.)