Justify your answers. Cross out what is not meant to be part of your final answer. Total number of points: 45.

I. (5 pts) Show that for any complex numbers $z_1$ and $z_2$,

$$||z_1| - |z_2|| \leq |z_1 + z_2|.$$
II. (5 pts) Find all possible values of
\((-\sqrt{3} + i)^{1/5} \).
III.

1. (3 pts) Can the function $u(x, y) = x^2 - y^2 - x - y$ be the REAL part of an analytic function $f(z) = u(x, y) + iv(x, y)$? **Hint:** You may use the Laplace equation, if you wish.

2. (5 pts) Determine all functions $v(x, y)$ such that $f(z) = u(x, y) + iv(x, y)$ is analytic.
3. (3 pts) Find explicitly as a function of $z$ the $f(z)$ such that

$$f(z) = u(x, y) + iv(x, y).$$
IV. (6 pts) Compute the line integral

$$\int_C \frac{(z^3 + z^2 + z + 1)}{z^4} \, dz$$

where $C$ is the LOWER half-circle centered at 0 joining $\frac{-1-i}{\sqrt{2}}$ and $\frac{1-i}{\sqrt{2}}$ in the positive (counterclockwise) sense.
V. Let
\[ f(z) = \frac{1}{(2 - z)(z + 3)}. \]

1. (2 pts) Write \( f(z) \) as a sum of fractions, i.e.,
\[ f(z) = \frac{A}{z - 2} + \frac{B}{z + 3}; \]

2. (3 pts) Explain whether it is possible to expand \( f(z) \) in Laurent (or Taylor) power series of:
   (i) \( z \), that converges in \( 0 \leq |z| < 3? \)
(ii) $z$, that converges in $3 < |z|$?

(iii) $z + 1$, that converges in $1 < |z + 1| < 4$?
3. (4 pts) Write the Laurent series expansion of $f(z)$ for $5 < |z - 2| < \infty$ as a power series of $(z - 2)$. 
VI. (6 pts) Let

\[ f(z) = \frac{1}{(z^2 + z)(z + 2)^3}. \]

Compute the integral of \( f(z) \) on the circles of center 1 and radii 1/2, 3/2, and 100, respectively.
VII. (3 pts) Determine where in the complex plane the following functions are analytic ($\bar{z}$ is the complex conjugate of $z$):

(i) $\frac{e^z}{\sin z}$

(ii) $z(\bar{z} + i)$

(iii) $e^{\frac{1}{z-1}}$
VIII. (3 pts-BONUS) Determine the constant $A$ so that the following function is analytic everywhere.

$$f(z) = \begin{cases} 
A \frac{\cosh z - 1}{z^2} & \text{if } z \neq 0 \\
1 & \text{if } z = 0.
\end{cases}$$