Other Integrals

\[ I = \int_0^{\infty} dx \frac{\sin x}{x} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\sin x}{x} \]

Method

1. let \( z = x \): \( f(z) = \frac{\sin(x)}{x} \) analytic everywhere.
2. close the path.

\[ I = \frac{1}{2} \int_{-\infty}^{\infty} dx \frac{1}{2i} \left( e^{ix} e^{-ix} \right) \frac{1}{x} = \frac{1}{4i} \left[ \int_{-\infty}^{\infty} \frac{e^{ix}}{x} - \int_{-\infty}^{\infty} dx \frac{e^{ix}}{x} \right] \]

\[ I = \frac{1}{2} \int_{C_1} dz \frac{\sin z}{z} \]

\[ = \frac{1}{4i} \left[ \int_{C_1} dz \frac{e^{iz}}{z} - \int_{C_2} dz \frac{e^{-iz}}{z} \right] \]

\[ I_{A} = \int_{C_1} dz \frac{e^{iz}}{z} = 0 \]

by Cauchy's Integral Formula

\[ C_{+} = C_{1} + C_{2}, \quad \int_{C_1} dz \frac{e^{iz}}{z} = 0 \]

\[ \int_{C_2} dz \frac{e^{-iz}}{z} \rightarrow \int_{A} = 0 \]

\[ I_{B} = \int_{C_2} dz \frac{e^{-iz}}{z} \]

\[ C_{-} = C_{2} + C_{R} \]

Residue theorem: \( \int_{C} dz \frac{e^{iz}}{z} = -2\pi i \), \( \frac{e^{z}}{z} \rightarrow -2\pi i \)

\[ \frac{1}{4\pi} \cdot 2\pi \approx \frac{\pi}{2} \]

Alternatively, \( I = \frac{1}{2} \lim_{k \to 0} \left( \int_{-\infty}^{k} + \int_{-\infty}^{k} \frac{\sin x}{x} \right) dx \)
\[ I = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\sin x}{x} \, dx \]

Method 2:

\[ I = \frac{1}{2} \lim_{\varepsilon \to 0} \left( \int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \right) \frac{\sin x}{x} \, dx = \frac{1}{2} \lim_{\varepsilon \to 0} \left( \int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \right) \frac{\sin x}{x} \, dx \]

\[ = \text{P} \int_{-\infty}^{\infty} \frac{\sin x}{x} \, dx \quad \text{principal value for } x = 0 \]

of \( \int_{-\infty}^{\infty} \frac{\sin x}{x} \, dx \)

\[ I = \frac{1}{2} \text{P} \int_{-\infty}^{\infty} \frac{\sin x}{x} \, dx = \frac{1}{2} \text{Im} \left( \text{P} \int_{-\infty}^{\infty} \frac{e^{ix}}{x} \, dx \right) \]

\[ = \frac{1}{2} \text{Im} \left( \text{P} \int_{-\infty}^{\infty} + \int_{\varepsilon}^{\infty} \right) \frac{e^{ix}}{x} \, dx \]

\[ \text{well-defined} \]

\[ \lim_{\varepsilon \to 0} \text{Im} \left( \int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \right) \frac{e^{ix}}{x} \, dx \]

\[ \text{Residue Theorem: } \int_{C} \frac{e^{iz}}{z} \, dz = 0 = \left( \int_{C_{E}} + \int_{C_{R}} + \int_{C_{\infty}} \right) \frac{e^{iz}}{z} \, dz \]

\[ = \int_{C_{E}} \frac{e^{iz}}{z} \, dz + \int_{C_{R}} \frac{e^{iz}}{z} \, dz + \int_{C_{\infty}} \frac{e^{iz}}{z} \, dz \]

\[ C = C_{E} + C_{R} + C_{\infty} \]

\[ x \to z, \quad \frac{e^{iz}}{z} \to \frac{e^{iz}}{z} \]

\[ (e^{iz}, \, z = \pm \infty) \]

\[ C = C_{E} + C_{(E)} + C_{R}^{(\infty)} \]

\[ \text{Residue Theorem: } \int_{C} \frac{e^{iz}}{z} \, dz = 0 = \left( \int_{C_{E}} + \int_{C_{R}} + \int_{C_{\infty}} \right) \frac{e^{iz}}{z} \, dz \]

\[ \left| \int_{C_{R}} \frac{e^{iz}}{z} \, dz \right| \leq \int_{0}^{\pi} \text{Re} \left( e^{-\pi \sin \theta} \right) \, d\theta \]

\[ \int_{0}^{\pi} \text{Re} \left( e^{-\pi \sin \theta} \right) \, d\theta \]

\[ z = \text{Re} \]

\[ d\theta = \text{Im} \]

\[ = \int_{0}^{2\pi} \text{Re} \left( e^{-\pi \sin \theta} \right) \, d\theta \]

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\[ \leq 2 \int_{0}^{\pi} \text{Re} \left( e^{-\pi \sin \theta} \right) \, d\theta \]

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\[ = 2 \frac{\pi}{2} \text{Re} \left( e^{-\pi \sin \theta} \right) \]

\[ \text{integral} \text{ goes to } 0 \text{ not exponentially, but as } \frac{1}{k} \]
\[
\int_{C_{e1}} \frac{dz}{iz} = \int_{C_{e2}} \frac{e^{iz}}{iz} \quad (\text{for } z = e^i\pi) \\
\int_{C_{e1}} \frac{e^{iz}}{iz} \quad (\text{for } z = e^{i\pi}) \\
0 = -i\pi + \int_{C_{e1}} \frac{e^{iz}}{iz} \\
\int_{C_{e1}} \frac{dz}{iz} = 1\pi \\
I = \frac{1}{2} \text{Im} \int_{C_{e1}} \frac{e^{iz}}{iz} = \frac{\pi}{2} \\
I = \Re \left( \int_{-\infty}^{\infty} dx \frac{\cos x}{x^2 + \frac{1}{4}} \right)
\]

\[
I = \pi \int_{-\infty}^{\infty} dx \frac{\cos x}{x^2 + \frac{1}{4}} \\
= \pi \left( \int_{-\infty}^{\frac{\pi}{4}} + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} + \int_{\frac{\pi}{2}}^{\infty} dx \frac{\cos x}{x^2 + \frac{1}{4}} \right)
\]

\[
I = \Re \left( \int_{-\infty}^{\infty} dx \frac{e^{ix}}{x^2 + \frac{1}{4}} \right)
\]

\[
I = \int_{C_{e1}} \frac{e^{iz}}{iz} = 0 \\
= \left( \int_{\frac{\pi}{2}}^{\pi} + \int_{\pi}^{\frac{3\pi}{2}} + \int_{\frac{3\pi}{2}}^{\pi} \right) dz \frac{e^{iz}}{iz} \\
= \frac{1}{2} e^{i\pi} + e^{2i\pi} + e^{3i\pi} = \frac{1}{2} + 1 + 1 = 2 \\
\]

\[
\int_{C_{e2}} \frac{e^{iz}}{iz} = -2 \\
I = -2
\]

\[
\int_{-\infty}^{\infty} dx \frac{\cos x}{x^2 + \frac{1}{4}} = \frac{1}{\pi} \left[ \int_{-\infty}^{\infty} dx \frac{\cos x}{x^2 + \frac{1}{2}} + \int_{-\infty}^{\infty} dx \frac{\cos x}{x^2 + \frac{3}{2}} \right]
\]

\[
\int_{-\infty}^{\infty} dx \frac{\cos x}{x^2 + \frac{1}{2}} = \frac{1}{2} \left[ \int_{-\infty}^{\infty} dx \frac{\cos x}{x^2 + \frac{1}{2}} + \int_{-\infty}^{\infty} dx \frac{\cos x}{x^2 + \frac{3}{2}} \right]
\]

\[
1 + 1 = 2 \\
1 + 1 = 2
\]