18.085 Computational Science and Engineering I
Fall 2008

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Problem 1 (40 points)

This question is about a fixed-free hanging bar (made of 2 materials) with a point load at \( x = \frac{3}{4} \):

\[
\begin{align*}
-\frac{d}{dx} \left( c(x) \frac{du}{dx} \right) &= \delta \left( x - \frac{3}{4} \right) \\
u(0) &= 0 \\
w(1) &= 0
\end{align*}
\]

Suppose that

\[
c(x) = \begin{cases} 
1, & x < \frac{1}{2} \\
4, & x > \frac{1}{2}
\end{cases}
\]

a) (i) At \( x = \frac{1}{2} \), \( u \) and \( w \) are continuous. Then \( u_x \) must have a jump (ii) At \( x = \frac{3}{4} \), \( u \) is continuous (as always) while \( w \) jumps by 1. We should expect \( \frac{dw}{dx} \) to have a jump unless such jump is "accidentally" 0.

b) \( w(x) = \begin{cases} 
A, & 0 < x < \frac{3}{4} \\
B, & \frac{3}{4} < x < 1
\end{cases} \)

where the three constants \( A \), \( B \), and \( C \) are determined from the boundary condition \( w(1) = 0 \), resulting in

\[
C = 0,
\]

continuity of \( w \) at \( x = \frac{1}{2} \), resulting in

\[
4B - A = 0,
\]

and \( [w]_- = -1 \), resulting in

\[
4C - 4B = -1
\]

This system with three equation and three unknowns is easily solved, yielding \( A = 1, B = \frac{1}{4}, C = 0 \). Summarizing:

\[
w(x) = \begin{cases} 
1, & 0 < x < \frac{3}{4} \\
\frac{1}{4}, & \frac{3}{4} < x < 1
\end{cases}
\]

c) \( u = \begin{cases} 
x + D, & 0 < x < \frac{1}{4} \\
\frac{1}{4}x + E, & \frac{1}{4} < x < \frac{3}{4} \\
F, & \frac{3}{4} < x < 1
\end{cases} \)

where the three constants \( D \), \( E \), and \( F \) are determined from the boundary condition \( u(0) = 0 \):

\[
D = 0,
\]
continuity of \( u \) at \( x = \frac{1}{2} \):

\[
\frac{1}{4} \times \frac{1}{2} + E - \frac{1}{2} - D = 0,
\]

and continuity of \( u \) at \( x = \frac{3}{4} \):

\[
F - \frac{1}{4} \times \frac{3}{4} - E = 0.
\]

We find that \( D = 0, E = \frac{3}{8}, F = \frac{9}{16} \) and so

\[
u = \begin{cases}
\frac{1}{4}x + \frac{3}{8}, & 0 < x < \frac{1}{2} \\
\frac{9}{16}, & \frac{1}{2} < x < 1
\end{cases}
\]

**Problem 2 (30 points)**

a)

(i) It is easy to show that

\[
\left( \frac{x}{x^2 + y^2} - i \frac{y}{x^2 + y^2} \right) (x + iy) = 1
\]

Therefore,

\[
\frac{1}{z} = \frac{x}{x^2 + y^2} - i \frac{y}{x^2 + y^2}
\]

and the real and imaginary parts are

\[
u(x, y) = \frac{x}{x^2 + y^2} \quad \text{ and } \quad s(x, y) = -\frac{y}{x^2 + y^2}
\]

(ii) In polar coordinates we have

\[
\frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r} e^{-i\theta} = \frac{1}{r} (\cos \theta - i \sin \theta)
\]

Therefore,

\[
u(r, \theta) = \frac{1}{r} \cos \theta \quad \text{ and } \quad s(r, \theta) = -\frac{1}{r} \sin \theta
\]

b) The curve \( u(x, y) = \frac{1}{2} \) has the following equation:

\[
\frac{x}{x^2 + y^2} = \frac{1}{2}
\]

This equation is equivalent to

\[
x^2 + y^2 - 2x = 0
\]
or
\[ x^2 - 2x + 1 + y^2 = 1 \]
or, finally
\[ (x - 1)^2 + y^2 = 1 \]
Similarly, the curve \( s(x, y) = \frac{1}{2} \) is given by
\[ x^2 + (y + 1)^2 = 1 \]
The curve \( u(x, y) = \frac{1}{2} \) is a circle of radius 1 centered at the point \((1, 0)\) while the curve \( s(x, y) = \frac{1}{2} \) is a circle of radius 1 centered at \((0, -1)\).

(c) On the part \( u(x, y) = \frac{1}{2} \) simply take \( u_0 = \frac{1}{2} \). Now let's look at the other part. It is orthogonal to the equipotentials of \( u \). In other words, \( u \) does not change in the directions orthogonal to the part. Analytically, this is expressed as \( \frac{\partial s}{\partial n} = 0 \) or \( w \cdot n = 0 \).

**Problem 3 (30 points)**

a). We have

\[
\begin{align*}
  u_x &= \frac{\partial^2 F}{\partial y \partial x} \\
  u_y &= \frac{\partial^2 F}{\partial y^2} \\
  s_x &= \frac{\partial^2 F}{\partial x^2} \\
  s_y &= \frac{\partial^2 F}{\partial x \partial y}
\end{align*}
\]

It can be immediately observed that \( u_x = s_y \) since partial derivatives commute. Also, \( u_y + s_x = \Delta F = 0 \) since \( F \) is harmonic.

b). The test for "having originated from a potential" is "\( \text{curl} u = 0 \)". Reconstructing the potential is a little less straightforward.

(i) \( v(x, y) = (x^2, y^2) \): Yes, \( u = \frac{1}{3}x^3 + \frac{1}{3}y^3 \), \( \Delta u = 2x + 2y \neq 0 \)

(ii) \( v(x, y) = (y^2, x^2) \): No.

(iii) \( v(x, y) = (x + y, x - y) \): Yes, \( u = \frac{1}{2}x^2 + xy - \frac{1}{2}y^2 \), \( \Delta u = 0 \)

c) (i)

\[
  u(r, \theta) = \frac{1}{2} + r \cos \theta + r^2 \cos 2\theta
\]

(ii)

\[
  u(r = 0, \theta) = \frac{1}{2}
\]

(A harmonic function equals to the average of its neighbors!)

\[
  u \left( r = \frac{1}{2}, \theta = 0 \right) = \frac{5}{4}
\]
Miscellaneous

Problem 1d) This is what the solution would be if we were to account for the weight $P$ of the bottom part of the bar

$$w(x) = \begin{cases} 
G, & 0 < x < \frac{3}{4} \\
-Px + H, & \frac{1}{2} < x < \frac{3}{4} \\
-Px + I, & \frac{3}{4} < x < 1
\end{cases}$$

The same three conditions will determine the constants: $w(1) = 0$:

$$I - P = 0,$$

continuity of $w$ at $x = \frac{1}{2}$:

$$4 \left( -\frac{1}{2}P + H \right) - G = 0,$$

and the jump in $w$ at $x = \frac{3}{4}$:

$$4I - 4H = -1$$

The resulting system determines the unknown constants: $G = 6P + 1$, $H = P + \frac{1}{4}$, and $I = P$:

$$w(x) = \begin{cases} 
6P + 1, & 0 < x < \frac{3}{4} \\
P(1-x) + \frac{1}{4}, & \frac{1}{2} < x < \frac{3}{4} \\
P(1-x), & \frac{3}{4} < x < 1
\end{cases}$$