1) (30 pts.) A system with 2 springs and masses is fixed-free. Constants are $c_1, c_2$.

(a) Write down the matrices $A$ and $K = A^TCA$.

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \quad K = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}$$

(b) Prove by two tests (pivots, determinants, independence of columns of $A$) that this matrix $K$ is (positive definite) (positive semidefinite).

Tell me which two tests you are using!

Determinants of $K$: $c_1 + c_2$ (1 x 1) and $c_1c_2$ (2 x 2)

Pivots of $K$: $c_1 + c_2$ and $c_1c_2/(c_1 + c_2)$

Independence of columns of $A$: $(1, -1)$ and $(0, 1)$

All prove positive definiteness of $K$.

(c) Multiply column times row to compute the “element matrices” $K_1, K_2$:

Compute $K_1 = (\text{column 1 of } A^T)(c_1)(\text{row 1 of } A)$

$$= c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Compute $K_2 = (\text{column 2 of } A^T)(c_2)(\text{row 2 of } A)$

$$= c_2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Then $K = K_1 + K_2$. What vectors solve $K_2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$?

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} C \\ C \end{bmatrix}$$

For those displacements $x_1$ and $x_2$, what is the energy in spring 2?

Zero (no stretching!)
2) (33 pts.) A network of nodes and edges and their conductances \( c_i > 0 \) is drawn without arrows. Arrows don’t affect the answers to this problem; the edge numbers are with the \( c \)'s. Node 5 is grounded (potential \( u_5 = 0 \)).

(a) List all positions \((i, j)\) of the 4 by 4 matrix \( K = A^TCA \) that will have zero entries. What is row 1 of \( K \)?

No bars node 1 to node 3, node 1 to node 4 (and 3 to 5). So \( K_{13} = K_{31} = K_{14} = K_{41} = 0 \) (and \( K_{\text{unreduced}} \) would have \( K_{35} = K_{53} = 0 \): not asked).

Row 1 of \( K \) comes from bar 1: \([c_1 + c_3, -c_1, 0, 0]\)

(b) Find as many independent solutions as possible to Kirchhoff’s Law \( A^Ty = 0 \).

Here we need arrows (sorry) to give consistent signs:

\[
\begin{bmatrix}
1 \\
0 \\
-1 \\
1 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
; \quad
\begin{bmatrix}
0 \\
0 \\
0 \\
-1 \\
1 \\
0 \\
1 \\
1
\end{bmatrix}
; \quad
\begin{bmatrix}
0 \\
1 \\
0 \\
0
\end{bmatrix}
\]

(c) Is \( A^TA \) always positive definite for every matrix \( A \)?

No. (\( A \) is any matrix)

If there is a test on \( A \), what is it?

(\( A \) must have independent columns. It can be tall and thin!)

What is the trick that proves \( u^TKu \geq 0 \) for every vector \( u \)?

\[
u^TKu = (u^TA^T)C(Au) = e^TCe = c_1e_1^2 + \cdots + c_me_m^2.
\]
3) **(37 pts.)** Make the network in Problem 2 into a 7-bar truss! The grounded node 5 is now a supported (but turnable) pin joint, with known displacements \( u_5^H = u_5^V = 0 \). **All angles are 45° or 90°.**

(a) How many rows and columns in the (reduced) matrix \( A \), after we know \( u_5^H = u_5^V = 0 \)?

7 rows (7 bars) and 8 columns (8 unknown \( u \)'s).

Describe in words (or a picture) all solutions to \( Au = 0 \).

\( Au = 0 \) when \( u \) = rigid rotation around node 5. (\( A \) has a 1-dimensional nullspace.)

If you add 1 bar can \( A \) become square and invertible?

Not invertible since rotation is still allowed.

(b) Write out row 2 of \( A \), corresponding to bar 2.

Row 2 = [0 0 \(-\sqrt{2}/2 \) \( \sqrt{2}/2 \) \( \sqrt{2}/2 \) \(-\sqrt{2}/2 \) 0 0]

Then (row 2) times the column \( u \) of displacements has what physical meaning?

(Row 2)\( u \) is the infinitesimal stretching of bar 2 in response to the small displacements \( u \).

(c) What is the first equation of \( A^T w = f \) (with right side \( f_1^H \))?

The first equation is the horizontal force balance at node 1. Since \( y \) measures stretching (rather than compression) according to our convention, the horizontal force balance at node 1 is \( y_1 = -f_1^H \).

Why does \( \frac{1}{2} u^T K u = \frac{1}{2} y^T C^{-1} y \) and what does this quantity represent physically?

\( \frac{1}{2} u^T K u = \frac{1}{2} e^T Ce = \frac{1}{2} y^T C^{-1} y \) represents the internal energy in the 7 bars.