1) (36 pts.) (a) For \(-\frac{d^2u}{dx^2} = \delta(x - a)\) with \(u(0) = u(1) = 0\), the solution is linear on both sides of \(x = a\) (graph = triangle from two straight lines). What is wrong with the next sentence? Integrating both sides of the equation from \(x = 0\) to \(x = 1\), the area under the triangle graph is 1. The base is 1 so the height must be \(u_{\text{max}} = 2\).

(b) If \(u(x, a)\) is the true solution to that standard problem in part (a), and the load point \(x = a\) approaches \(x = 1\), what function does the solution \(u\) approach? Give a physical reason for your answer or a math reason or both.
xxx
2) (40 pts.) Which of these 5 equations can be solved?? If the equation has a solution, please find one. If not why not??

(a) \[ \text{div} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix} = 1 \]

(b) \[ \text{div} \begin{bmatrix} \frac{\partial s}{\partial y} \\ -\frac{\partial s}{\partial x} \end{bmatrix} = 1 \]

(c) \[ u_{xx} + u_{yy} = 0 \] in the unit circle and \( u(1, \theta) = \sin 4\theta \) around the boundary

(d) Find a family of curves \( u(x, y) = C \) that is everywhere perpendicular to the family of curves \( x + x^2 - y^2 = C \).

(e) \[ \frac{d^4 u}{dx^4} = \delta(x) \] [point load at \( x = 0 \), not requiring boundary conditions, any solution \( u(x) \) is OK].
xxx
3) **(24 pts.)** I want to solve Laplace’s equation (really Poisson’s equation) in 3D with no boundaries (the whole space). The right side is a point load $\delta$ at the origin $(0,0,0)$. So $-\text{div}(\text{grad } u) = \delta$.

(a) Integrate both sides over a sphere of radius $R$ around the origin:

$$\iiint -\text{div}(\text{grad } u) \, dxdydz = 1.$$  

Use the divergence theorem (or Gauss-Green) to transform that triple integral into an integral on the surface of the sphere of radius $R$.

(b) What is the normal vector $n$ in that integral? Knowing that $u$ must be radially symmetric, $\frac{\partial u}{\partial r}$ is a constant on the sphere of radius $R$. What is that constant?

(c) So what is $u(r)$?
xxx