1) (36 pts.) (a) For $-\frac{d^2u}{dx^2} = \delta(x - a)$ with $u(0) = u(1) = 0$, the solution is linear on both sides of $x = a$ (graph = triangle from two straight lines). What is wrong with the next sentence? Integrating both sides of $-u'' = \delta(x-a)$, from $x = 0$ to $x = 1$, the area under the triangle graph is 1. The base is 1 so the height must be $u_{\text{max}} = 2$.

(b) If $u(x,a)$ is the true solution to that standard problem in part (a), and the load point $x = a$ approaches $x = 1$, what function does the solution $u$ approach? Give a physical reason for your answer or a math reason or both.

Integrating $\delta(x - a)$ does give 1. But the area under the graph of $u(x)$ is $\int u(x) \, dx$ and not $\int -u''(x) \, dx$. (The integral of $-u''(x) = \delta(x - a)$ gives the drop in slope $u'(0) - u'(1) = 1$.) So the reasoning is wrong and $u_{\text{max}}$ is not 2. The actual $u_{\text{max}}$ is $(a - 1) a$, because the true solution has slope $a - 1$ up to the load point $x = a$. As that point approaches $a = 1$, the solution $u(x)$ approaches zero. Physically, the load is moving close to the support. Then the load causes a smaller and smaller displacement.
2) (40 pts.) Which of these 5 equations can be solved?? If the equation has a solution, please find one. If not why not??

(a)
\[
\text{div} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix} = 1
\]

(b)
\[
\text{div} \begin{bmatrix} \frac{\partial s}{\partial y} \\ -\frac{\partial s}{\partial x} \end{bmatrix} = 1
\]

(c) \(u_{xx} + u_{yy} = 0\) in the unit circle and \(u(1, \theta) = \sin 4\theta\) around the boundary

(d) Find a family of curves \(u(x,y) = C\) that is everywhere perpendicular to the family of curves \(x + x^2 - y^2 = C\).

(e) \(d^4u/dx^4 = \delta(x)\) [point load at \(x = 0\), not requiring boundary conditions, any solution \(u(x)\) is OK].

(a) This is Poisson’s equation \(u_{xx} + u_{yy} = 1\). One solution is \(u(x,y) = \frac{1}{2}x^2\).

(b) This equation is \(\frac{\partial^2 s}{\partial y \partial x} - \frac{\partial^2 s}{\partial x \partial y} = 1\). No solution since \(s_{xy} = s_{yx}\).

(c) \(u(r, \theta) = r^4 \sin 4\theta\) solves Laplace’s equation and reduces to \(\sin 4\theta\) on the unit circle \(r = 1\).

(d) The function \(x + x^2 - y^2\) is the real part of \((x + iy) + (x + iy)^2\). So we get perpendicular curves from the imaginary part \(y + 2xy = C\).

(e) \(u = \begin{cases} 
0 & \text{for } x \leq 0 \\
\frac{x^3}{6} & \text{for } x \geq 0 
\end{cases}\) has jump of 1 in \(u''\) and \(u''' = \delta(x)\). It comes from integrating \(\delta(x)\) four times.
3) (24 pts.) I want to solve Laplace’s equation (really Poisson’s equation) in 3D with no boundaries (the whole space). The right side is a point load $\delta$ at the origin $(0,0,0)$. So $-\text{div}(\text{grad } u) = \delta$.

(a) Integrate both sides over a sphere of radius $R$ around the origin:

$$\int \int \int -\text{div}(\text{grad } u) \, dx\,dy\,dz = 1.$$ 

Use the divergence theorem (or Gauss-Green) to transform that triple integral into an integral on the surface of the sphere of radius $R$.

(b) What is the normal vector $n$ in that integral? Knowing that $u$ must be radially symmetric, $\frac{\partial u}{\partial r}$ is a constant on the sphere of radius $R$. What is that constant?

(c) So what is $u(r)$?

(a) The divergence theorem transforms to a double integral for the flux through the sphere:

$$\int \int (\text{grad } u) \cdot n \, dS = 1.$$ 

(b) Since $n$ is the outward (radial) unit vector, $(\text{grad } u) \cdot n$ is the same as $\frac{\partial u}{\partial r}$. It is constant on the sphere (which has area $4\pi R^2$) so its value is $1/4\pi R^2$.

(c) If $\frac{\partial u}{\partial r} = 1/4\pi r^2$ whenever $r = R$, then $u(r) = 1/4\pi r$. 