1) (36 pts.) The 5 nodes in the network are at the corners of a *square* and the center. Node 5 is grounded so $x_5 = 0$. All 8 edges have conductances $c = 1$ so $C = I$.

(a) Fill in the 8 by 4 incidence matrix $A$ (node 5 grounded). What is $A^T A$? Is $A^T A$ invertible (YES, NO)?

\[
A_{8 \times 4} = \begin{bmatrix}
-1 & 1 & 0 & 0 \\
-1 & 0 & 1 & 0 \\
-1 & 0 & 0 & 1 \\
0 & -1 & 1 & 0 \\
0 & 0 & -1 & 1 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix}
\]

\[
A^T A = \begin{bmatrix}
3 & -1 & -1 & -1 \\
-1 & 3 & -1 & 0 \\
-1 & -1 & 4 & -1 \\
-1 & 0 & -1 & 3
\end{bmatrix}
\]
(b) How many independent solutions to $A^Ty = 0$? Write down one nonzero solution $y$.

**Ans.** The upper left loop gives $y = (1, -1, 0, 1, 0, 0, 0, 0)$

(c) The current source $f_1 = 3$ enters node 1 and exits at grounded node 5. In 2 by 2 block form (using $A$), what are the 12 equations for the 8 currents $y$ and the 4 potentials $x$?

\[
\begin{bmatrix}
I & A \\
A^T & 0
\end{bmatrix}
\begin{bmatrix}
y \\
x
\end{bmatrix}
= 
\begin{bmatrix}
b \\
f
\end{bmatrix}
\text{ with } \begin{bmatrix}
b \\
f
\end{bmatrix} = \begin{bmatrix}
3 \\
0 \\
0 \\
0
\end{bmatrix}
\]

(d) Write out in full with numbers the 4 equations for the 4 potentials, after the currents $y$ are eliminated. Using symmetry (or guessing or solving) what is the solution $x_1, x_2, x_3, x_4$?

**Ans.** The equations are $A^T Ax = f$ and the solution is $x = (2, 1, 1, 1)$. Unit currents flow to $x_5$ on edges 1–6 and 2–7 and 3–8. Voltage drop $= 1$ on those six edges.
2) (24 pts.) The same 8 edges and 5 nodes form a square pin-jointed truss. The pin at node 5 is held in position so \( x_5^H = x_5^V = 0 \). All 8 elastic constants are \( c = 1 \) so \( C = I \).

(a) How many unknown displacements? \( \mathbf{8} \)

What is the shape of the matrix \( A \) in \( e = Ax \)? \( \mathbf{8 \ by \ 8} \)

Find the first column of \( A \), corresponding to the stretching \( e \) in the 8 edges from a small displacement \( x_1^H \) at node 1.

\[
\begin{bmatrix}
-\sqrt{2}/2 \\
0 \\
+\sqrt{2}/2 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

(b) Are there any nonzero solutions to \( Ax = 0 \)? (YES, NO)

How many independent solutions do you physically expect? \( \mathbf{1} \)

Draw a picture of each independent solution (if any) to show the movement of the 4 nodes.

**Ans.** Rotation around node 5 has \( x = (2, 0, 1, 1, 0, 1, -1) \).
(c) How many independent solutions to $A^T y = 0$? Can you find them?

**Ans.** Since $A$ is square, there will be one line of solutions to $A^T y = 0$ when there is one line of solutions to $A x = 0$ (Rank 7). The equations $A^T y = 0$ look for a set of bar forces that balance themselves! One set is drawn here:
3) **(40 pts.)**

(a) Find a 4th degree polynomial \( s(x, y) \) with only 2 terms that solves Laplace’s equation. Please draw a box around your answer \( s(x, y) \).

**Ans.** \((x + iy)^4\) gives \( s(x, y) = 4x^3y - 4xy^3 \).

(b) In the \( xy \) plane draw all the solutions to \( s(x, y) = 0 \). Then in the same picture *roughly* draw the curve \( s(x, y) = c \) that goes through the particular point \((x, y) = (2, 1)\).

**Ans.** \( 4x^3y = 4xy^3 \) gives \( x = 0 \) or \( y = 0 \) or \( x = \pm y \) (four lines). Through \( x = 2, y = 1 \) will go the curve \( s(x, y) = 4 \cdot 8 - 4 \cdot 2 = 24 \). It won’t cross the lines because they have \( s(x, y) = 0 \). It will get close to the lines \( y = 0 \) and \( x = y \) as \( x \) gets large, because \( 4x^3y - 4xy^3 = 24 \) gives \( xy(x + y)(x - y) = 6 \). If \( x \) and \( x + y \) get large then either \( y \) or \( x - y \) must get small! The curve isn’t a hyperbola, I think it must be symmetric across the line \( \theta = \pi/8 \).

(c) If the curves \( s(x, y) = c \) are the *streamlines* of a potential flow (in the usual framework), what is the corresponding velocity \( v(x, y) = w(x, y) \)?

\[
 w(x, y) = \frac{\partial s}{\partial y} - \frac{\partial s}{\partial x} = (4x^3 - 12xy^2, 4y^3 - 12yx^2) .
\]

(d) (this Green’s formula question is *not* related to parts a, b, c)

Suppose \( w(x, y) = (w_1(x, y), 0) \) is a flow field. With \( w_2 = 0 \) write down the remaining (not zero) terms in Green’s formula for the integral \( \iint (\text{grad } u) \cdot \text{w } dx \, dy \) in the unit square \( 0 \leq x \leq 1, 0 \leq y \leq 1 \). Substitute for \( n \) and \( ds \) when you know what they are for this square.

**Ans.** Green’s formula in the plane is

\[
 \iint (\text{grad } u) \cdot \text{w } dx \, dy = - \iint u \text{ div } \text{w } dx \, dy + \int u \text{w } \cdot n \, ds .
\]
Here $w_2 = 0$ and $n = (1, 0)$ on the right side and $n = (-1, 0)$ on the left side. This leaves

$$
\int \int \frac{\partial u}{\partial x} w_1 \, dx \, dy = - \int \int u \frac{\partial w_1}{\partial x} \, dx \, dy + \int u w_1 \, dy - \int u \frac{\partial w_1}{\partial x} \, dy \, \text{up right side} - \int u \frac{\partial w_1}{\partial x} \, dy \, \text{up left side}
$$

(e) A one-dimensional formula on any horizontal line $y = y_0$ is integration by parts:

$$
\int_{x=0}^{1} \frac{du}{dx} w_1(x) \, dx = - \int_{x=0}^{1} u(x) \frac{dw_1}{dx} \, dx + u w_1(x = 1) - u w_1(x = 0).
$$

Here $u$ and $w_1$ are $u(x, y_0)$ and $w_1(x, y_0)$ since $y = y_0$ is fixed.

**Question 1**  How do you derive your Green’s formula in part (d) from this one-dimensional formula? ANSWER IN ONE SENTENCE, NO MATH SYMBOLS !!

**Ans.** Integrate the 1D formula from $y = 0$ to $y = 1$.

**Question 2** (not related)  Find all vector fields of this form $(w_1(x, y), 0)$ that can be velocity fields $v = w = (w_1(x, y), 0)$ in potential flow [so $v = \text{grad} \, u$ and div $w = 0$ as usual].

**Ans.** Potential flow with $w = (w_1(x, y), 0)$ requires

$$
\text{div} \, w = \frac{\partial w_1}{\partial x} = 0 \quad \text{and also} \quad w_1(x, y) = \frac{\partial u}{\partial x}.
$$

Then $w_1$ = constant! The only horizontal potential flow is uniform flow.