Your name is: ____________________________ Grading 1
2
3
Total

Thank you for taking 18.085, I hope you enjoyed it.

1) (35 pts.) Suppose the 2π-periodic $f(x)$ is a half-length square wave:

$$f(x) = \begin{cases} 
1 & \text{for } 0 < x < \pi/2 \\
-1 & \text{for } -\pi/2 < x < 0 \\
0 & \text{elsewhere in } [-\pi, \pi] 
\end{cases}$$

(a) Find the Fourier cosine and sine coefficients $a_k$ and $b_k$ of $f(x)$.

(b) Compute $\int_{-\pi}^{\pi} (f(x))^2 \, dx$ as a number and also as an infinite series using the $a_k^2$ and $b_k^2$.

(c) DRAW A GRAPH of the integral $I(x) = \int_{0}^{x} f(t) \, dt$ from $-\pi$ to $\pi$.

What are the Fourier coefficients $A_k$ and $B_k$ of $I(x)$?

(d) DRAW A GRAPH of the derivative $D(x) = \frac{df}{dx}$ from $-\pi$ to $\pi$. What are the Fourier coefficients of $D(x)$?

(e) If you convolve $D(x) * I(x)$ why do you get the same answer as $f(x) * f(x)$? Not required to find that answer, just explain $D * I = f * f$. 
2) \textbf{(33 pts.)} (a) Compute directly the convolution \( f \ast f \) (cyclic convolution with \( N = 6 \)) when \( f = (0, 0, 0, 1, 0, 0) \). [You could connect vectors \((f_0, \ldots, f_5)\) with polynomials \( f_0 + f_1w + \cdots + f_5w^5 \) if you want to.]

(b) What is the Discrete Fourier Transform \( c = (c_0, c_1, c_2, c_3, c_4, c_5) \) of the vector \( f = (0, 0, 0, 1, 0, 0) \)? Still \( N = 6 \).

(c) Compute \( f \ast f \) another way, by using \( c \) in “transform space” and then transforming back.
3) (32 pts.) On page 310 the Fourier integral transform of the one-sided decaying pulse
\[ f(x) = e^{-ax} \text{ (for } x \geq 0 \text{ only)} \]
is computed for \(-\infty < k < \infty\) as
\[ \hat{f}(k) = \frac{1}{a + ik}. \]

(a) Suppose this one-sided pulse is shifted to start at \(x = L\):
\[ f_L(x) = e^{-a(x-L)} \text{ for } x \geq L, \quad f_L(x) = 0 \text{ for } x < L. \]
Find the Fourier integral transform \(\hat{f}_L(k)\).

(b) Draw a rough graph of the difference \(D(x) = F(x) - F_L(x)\) and find its transform \(\hat{D}(k)\). NOW LET \(a \to 0\).

What is the limit of \(D(x)\) as \(a \to 0\)?

What is the limit of \(\hat{D}(k)\) as \(a \to 0\)?

(c) The function \(f_L(x)\) is smooth except for a ______ at \(x = L\), so the decay rate of \(\hat{f}_L(k)\) is ______ . The convolution \(C(x) = f_L(x) * f_L(x)\) has transform \(\hat{C}(k) = \ldots\) with decay rate ______ . Then in \(x\)-space this convolution \(C(x)\) has a ______ at the point \(x = \ldots\).
xxx