Thank you for taking 18.085, I hope you enjoyed it.

1) (35 pts.) Suppose the $2\pi$-periodic $f(x)$ is a half-length square wave:

$$f(x) = \begin{cases} 
1 & \text{for } 0 < x < \pi/2 \\
-1 & \text{for } -\pi/2 < x < 0 \\
0 & \text{elsewhere in } [-\pi, \pi]
\end{cases}$$

(a) Find the Fourier cosine and sine coefficients $a_k$ and $b_k$ of $f(x)$.

(b) Compute $\int_{-\pi}^{\pi} (f(x))^2 \, dx$ as a number and also as an infinite series using the $a_k^2$ and $b_k^2$.

(c) DRAW A GRAPH of its integral $I(x)$. (Then $dI/dx = f(x)$ on the interval $[-\pi, \pi]$ choose the integration constant so $I(0) = 0$.) What are the Fourier coefficients $A_k$ and $B_k$ of $I(x)$?

(d) DRAW A GRAPH of the derivative $D(x) = \frac{df}{dx}$ from $-\pi$ to $\pi$. What are the Fourier coefficients of $D(x)$?

(e) If you convolve $D(x) \ast I(x)$ why do you get the same answer as $f(x) \ast f(x)$? Not required to find that answer, just explain $D \ast I = f \ast f$.

(a) $f(x)$ is an odd function $= -f(-x)$ so all $a_k = 0$.

Half-interval: $b_k = \frac{2}{\pi} \int_0^{\pi/2} \sin kx \, dx = \frac{2}{\pi} \frac{1-\cos(k\pi/2)}{k}$. 
(b) $\int_{-\pi}^{\pi} (f(x))^2 \, dx = \int_{-\pi/2}^{\pi/2} = \pi$. By Parseval this equals $\pi \sum b_k^2$. (Substituting $b_k = \frac{2}{\pi} \left( \frac{1}{1}, \frac{2}{2}, \frac{3}{3}, \ldots \right)$ will give a remarkable formula from $\sum b_k^2 = 1$.)

Integration starting at 0 or $-\pi$

Even function so $B_k = 0$.

Integrating $b_k \sin kx$ gives $-b_k \cos kx$ so $A_k = -\frac{b_k}{k}$.

The constant term is $A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} I(x) \, dx = \frac{3\pi}{8} \text{ or } -\frac{\pi}{8}$ (integrate starting at 0 or $-\pi$).

(d) $D(x) = \frac{d}{dx} 2\delta(x)$

Even function so $B_k = 0$.

Derivative of $b_k \sin kx$ is $k b_k \cos kx$ so $A_k = k b_k$.

Constant term is $A_0 = 0$.

(e) Convolution in $x$-space is multiplication in $k$-space. So $f \ast f$ has complex Fourier coefficients $c_k^2$ (with factor $2\pi$). And $D(x) \ast I(x)$ has Fourier coefficients $(ik c_k)(c_k/i) = c_k^2$ (with same factor). $D \ast I = f \ast f!!$ Check in $x$-space:

$$\int_{-\pi}^{\pi} I(t) D(x - t) \, dt = \text{integrate by parts = }$$

$$\int_{-\pi}^{\pi} f(t) f(x - t) \, dt + \text{(boundary term = 0 by periodicity)}.$$ 

The usual minus sign disappears because of 2nd minus sign: $\frac{d}{dt} D(x - t) = -f(x - t)$.

NOTE: I have now learned that we can’t just multiply sine coefficients $(k b_k)(-b_k/k)$ because that gives an unwanted minus sign as in $\int \sin t \sin(x - t) \, dt = -\pi \cos x$. 
2) (33 pts.)

(a) Compute directly the convolution \( f \ast f \) (cyclic convolution with \( N = 6 \)) when \( f = (0, 0, 0, 1, 0, 0) \). [You could connect vectors \((f_0, \ldots, f_5)\) with polynomials \( f_0 + f_1w + \cdots + f_5w^5 \) if you want to.]

(b) What is the Discrete Fourier Transform \( \mathbf{c} = (c_0, c_1, c_2, c_3, c_4, c_5) \) of the vector \( f = (0, 0, 0, 1, 0, 0) \)? Still \( N = 6 \).

(c) Compute \( f \ast f \) another way, by using \( \mathbf{c} \) in “transform space” and then transforming back.

With \( N = 6 \) the complex number \( w = e^{2\pi i/6} \) has \( w^3 = -1 \) and \( w^6 = 1 \).

(a) \( f = (0, 0, 0, 1, 0, 0) \) corresponds to \( w^3 \). Then \( f \ast f \) corresponds to \( w^6 \) which is 1. So \( f \ast f = (1, 0, 0, 0, 0, 0) \). (Also seen by circulant matrix multiplication.)

(b) The transform \( \mathbf{c} = F^{-1}f = \frac{1}{6} \mathbf{F}f = \frac{1}{6} \) (column of \( \mathbf{F} \) with powers of \( w^3 = -1 \)): Then \( \mathbf{c} = \frac{1}{6} (1, -1, 1, -1, 1, -1) \).

(c) The transform of \( f \ast f \) is \( \frac{6}{36} (1^2, (-1)^2, 1^2, (-1)^2, 1^2, (-1)^2) = \frac{1}{6} (1, 1, 1, 1, 1, 1) \).

Multiply that vector \( v \) by \( F \) to transform back and \( Fv = (1, 0, 0, 0, 0, 0) \) as in part (a)!
3) (32 pts.) On page 310 Example 3, the Fourier integral transform of the one-sided decaying pulse \( f(x) = e^{-ax} \) (for \( x \geq 0 \)) \( f(x) = 0 \) (for \( x < 0 \)) is computed for \( -\infty < k < \infty \) as

\[
\hat{f}(k) = \frac{1}{a + ik}.
\]

(a) Suppose this one-sided pulse is shifted to start at \( x = L > 0 \):

\[
f_L(x) = e^{-a(x-L)} \text{ for } x \geq L, \quad f_L(x) = 0 \text{ for } x < L.
\]

Find the Fourier integral transform \( \hat{f}_L(k) \).

(b) Draw a rough graph of the difference \( D(x) = f(x) - f_L(x) \), on the whole line \( -\infty < x < \infty \). Find its transform \( \hat{D}(k) \). NOW LET \( a \to 0 \).

What is the limit of \( D(x) \) as \( a \to 0 \)?

What is the limit of \( \hat{D}(k) \) as \( a \to 0 \)?

(c) The function \( f_L(x) \) is smooth except for a \( \boxed{\text{jump}} \) at \( x = L \), so the decay rate of \( \hat{f}_L(k) \) is \( \boxed{1/k} \). The convolution \( C(x) = f_L(x) * f_L(x) \) has transform \( \hat{C}(k) = \frac{e^{-ikL}}{(a + ik)^2} \) with decay rate \( \boxed{1/k^2} \).

Then in \( x \)-space this convolution \( C(x) \) has a \( \boxed{\text{corner (= ramp)}} \) at the point \( x = \boxed{2L} \).

(a) \( f_L(x) \) is \( f(x - L) \). By the shift rule (page 317) \( \hat{f}_L(k) = e^{-ikL} \hat{f}(k) = \frac{e^{-ikL}}{a + ik} \).

(b) \( D(x) \)

\[ 
\begin{array}{c}
D(x) \\
\hline
0 & L \\
\hline
\end{array}
\]

As \( a \to 0 \), \( D(x) \) approaches 1 for \( 0 < x < L \), 0 elsewhere

\[ a = 1: \text{Graph } e^{-x} \text{ then } e^{-x} - e^{-(x-L)} \]

\[ \hat{D}(k) = \frac{1}{a + ik} - \frac{e^{-ikL}}{a + ik} \text{ approaches } \frac{1-e^{-ikL}}{ik} = \text{transform of square pulse.} \]

(c) FILLED IN BLANKS ABOVE