1) (30 pts.) (a) Suppose $f(x)$ is a periodic function:

$$f(x) = \begin{cases} 
0 & \text{for } -\pi < x < 0 \\
e^{-x} & \text{for } 0 \leq x \leq \pi \\
f(x + 2\pi n) & \text{for every integer } n
\end{cases}$$

Find the coefficients $c_k$ in the complex Fourier series $f(x) = \sum c_k e^{ikx}$.

What is $c_0$? What is $\sum_{-\infty}^{\infty} |c_k|^2$?

(b) Draw a graph of $f(x)$ from $-2\pi$ to $2\pi$. Also draw a careful graph of $df/dx$. How quickly do the coefficients of $f(x)$ decay as $k \to \infty$ and why?

(c) Find the Fourier coefficients $d_k$ of $df/dx$. Do they approach a constant (or what pattern do they approach) as $k \to \infty$? Explain the pattern from your graphs.
Solution.

(a) \( c_k = \frac{1}{2\pi} \int_0^\pi e^{-x} e^{-ikx} \, dx = \frac{1}{2\pi} \frac{e^{-(1+ik)x} \bigg|_0^\pi}{-(1+ik)} = \frac{1}{2\pi} \frac{1 - e^{-(1+ik)\pi}}{1 + ik} = \frac{1 - (-1)^k e^{-\pi}}{2\pi(1 + ik)} \)

\( c_0 = \frac{1 - e^{-\pi}}{2\pi} \sum |c_k|^2 = \frac{1}{2\pi} \int_0^\pi (e^{-x})^2 \, dx = \frac{1 - e^{-2\pi}}{4\pi} \)

(b) The graph of \( f(x) \) includes a jump of 1 at \( x = 0 \) and a drop of \( e^{-\pi} \) at \( x = \pi \). So \( df/dx \) includes \( \delta(x) - e^{-\pi} \delta(x - \pi) \). (Both function have \( e^{-x} \) from 0 to \( \pi \).)

The coefficients of \( f(x) \) decay like \( 1/k \) because of the two jumps.

(c) The coefficients of \( df/dx \) are

\[ d_k = ik \, c_k = \frac{ik}{2\pi(1 + ik)}(1 + (-1)^k e^{-\pi}). \]

As \( k \to \infty \) they do not approach a constant (which would be 1, coming from \( \delta(x) \)). Instead the limiting pattern alternates between \( 1 + e^{-\pi} \) and \( 1 - e^{-\pi} \), because \( f(x) \) has two jumps.
2) (33 pts.)  (a) Can you complete this 4-step MATLAB code to compute the cyclic convolution \( f \circledast g = h \)? I suggest \( \text{fhat}, \text{ghat}, \text{hhat} \) for their transforms.

1. \( \text{fhat} = \text{fft}(f) \)
2. \( \text{ghat} = \text{fft}(g) \)
3. \( \text{hhat} = \text{fhat} .* \text{ghat} \)
4. \( h = \text{ifft}(\text{hhat}) \)

(It is equally possible to start with the inverse discrete transform \( \text{ifft} \). The only difference will be a factor of \( N \) somewhere, which I forgive! If you don’t know MATLAB notation for commands 2, 3, 4 you can use words. MATLAB’s \text{fft}(f) and \text{ifft}(f) automatically determine the length of \( f \).)

(b) Suppose each of your quiz grades is a random variable (don’t know how I thought of this). The probability of grade \( j \) on each quiz \( (j = 0, \ldots, 100) \) is \( p_j \). The “generating function” for that quiz is \( P(z) = \sum p_j z^j \). What is the probability \( s_k \) that the sum of your grades on 2 quizzes is \( k \)? Give a nice formula for \( S(z) = \sum s_k z^k \).

(c) The chance of grade \( j = (70, 80, 90, 100) \) on one quiz is \( p = (.3, .4, .2, .1) \). What is the expected value (mean \( m \)) for the grade on that quiz? Show that this quiz average \( m \) agrees with \( dP/dz \) at \( z = 1 \). What are the probabilities \( s_k \) for the sum of two grades? Give numbers or a MATLAB code for the \( s_k \).
Solution.

(b) The two grades are $i$ and $j$ with probability $p_i p_j$. Looking at all pairs that add to $k$,

$$s_k = \sum_{i+j=k} p_i p_j = \sum p_i p_{k-i} \quad \text{and} \quad s = p \ast p.$$ 

The convolution rule (multiplying polynomials is convolution of coefficients) says that $S(z) = (P(z))^2$. 

*I should have worded this problem more clearly.*

(c) The expected value (the mean $m$) is

$$(.3)(70) + (.4)(80) + (.2)(90) + (.1)(100) = 81.$$ 

This is the derivative at $z = 1$ of

$$P(z) = (.3)z^{70} + (.4)z^{80} + (.2)z^{90} + (.1)z^{100}.$$ 

For the probabilities $s_k$, part (b) says that we have to convolve $p \ast p$. Noncyclic convolution is $\text{conv}(p, p)$ — or pad $p$ by extra zeros and use the cyclic code in part (a) — or compute $(3421)^2$ without carrying:

$$
\begin{array}{cccc}
3 & 4 & 2 & 1 \\
3 & 4 & 2 & 1 \\
3 & 4 & 2 & 1 \\
6 & 8 & 4 & 2 \\
12 & 16 & 8 & 4 \\
9 & 12 & 6 & 3 \\
\end{array}
\begin{array}{cccc}
9 & 24 & 28 & 22 \\
12 & 4 & 1 = \text{percentages adding to 100}
\end{array}$$
3) (37 pts.)  

(a) The hat function $H(x) = 1 - |x|$ for $-1 \leq x \leq 1$ has height 1 and area 1 and integral transform $\hat{H}(k) = (2 - 2 \cos k)/k^2$. Find the transform $\hat{R}(k)$ of the roof function $R(x)$:

$$R(x) = \text{box} + \text{hat} = 2 - |x| \quad \text{for} \ -1 \leq x \leq 1, \ 0 \text{ else.}$$

(b) What is the value of $\hat{R}(k)$ at $k = 0$ and how does this connect to the graph of the roof?

(c) Suppose $R(x)$ is the response of a sensor to a point source $\delta(x)$ at $x = 0$. The sensor is shift-invariant (shifted response when source is shifted). The output $F$ from a distributed source $U(x)$ is the convolution $F = R \ast U$. Describe how to find $U(x)$ if you know $F(x)$.

(d) There could be a difficulty with your solution method in part (c). That would arise if $\underline{\ldots} = 0$. For 1 point, does this difficulty appear in this example?
Solution.

(a) The box on $[-1, 1]$ has transform $(e^{ik} - e^{-ik})/ik = 2\sin k/k$. Then $R = \text{box} + \hat{\text{hat}}$ has

$$\widehat{R} = \widehat{\text{box}} + \widehat{\hat{\text{hat}}} = \frac{2\sin k}{k} + \frac{2 - 2\cos k}{k^2}$$

Note: The $1/k$ decay rate comes from the jumps in the box function. The $1/k^2$ terms come from corners in the hat.

(b) $\widehat{R}(0) = 3$ because the area under $R(x)$ is $\int_{-1}^{1} R(x) e^{0x} dx = 3$.

(c) Take transforms of $F = R \ast U$ to find $\widehat{F} = \widehat{R} \widehat{U}$. Then $\widehat{U} = \widehat{F}/\widehat{R}$. Invert this transform to find $U(x)$.

(d) There is a difficulty if $\widehat{R}(k) = 0$ for any frequencies $k$. This does appear in the example when $k = 2\pi, 4\pi, \ldots$