1) (39 pts.) With $h = \frac{1}{3}$ there are 4 meshpoints $0, \frac{1}{3}, \frac{2}{3}, 1$ and displacements $u_0, u_1, u_2, u_3$.

a) Write down the matrices $A_0, A_1, A_2$ with three rows that produce the first differences $u_i - u_{i-1}$:

$A_0$ has 0 boundary conditions on $u$
$A_1$ has 1 boundary condition $u_0 = 0$ (left end fixed)
$A_2$ has 2 boundary conditions $u_0 = u_3 = 0$.

b) Write down all three matrices $A_0^T A_0, A_1^T A_1, A_2^T A_2$.

CROSS OUT IF FALSE / GIVE REASON BASED ON COLUMNS OF $A$!

$K_0 = A_0^T A_0$ is (singular) (invertible) (positive definite)

*Reason:*

$K_1 = A_1^T A_1$ is (singular) (invertible) (positive definite)

*Reason:*

c) Find all solutions $w = (w_1, w_2, w_3)$ to each of these equations:

$$A_0^T w = 0 \quad A_1^T w = 0 \quad A_2^T w = 0$$
Solution.

a) 
\[ A_0 = \begin{bmatrix} -1 & 1 \\ -1 & 1 \\ -1 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 \\ -1 & 1 \\ -1 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \]

b) 
\[ A_0^T A_0 = \begin{bmatrix} 1 & -1 \\ -1 & 2 & -1 \\ -1 & 2 & -1 \\ -1 & 1 \end{bmatrix}, \quad A_1^T A_1 = \begin{bmatrix} 2 & -1 \\ -1 & 2 & -1 \\ -1 & 1 \end{bmatrix}, \quad A_2^T A_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \]

\[ \downarrow \quad \text{singular} \quad \downarrow \quad \text{(invertible) (positive definite)} \]

c) 
\[ A_0^T w = \begin{bmatrix} -1 \\ 1 & -1 \\ 1 & -1 \\ 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \rightarrow w = 0 \]
\[ A_1^T w = 0 \quad \rightarrow \quad w = 0 \]
\[ A_2^T w = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \rightarrow \quad w = c \begin{bmatrix} 1 \\ 1 \end{bmatrix} \]
2) (33 pts.) a) Find the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and unit eigenvectors $y_1, y_2, y_3$ of $B$.

Hint: one eigenvector is $(1, 0, -1)/\sqrt{2}$.

\[
B = \begin{bmatrix}
1 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 1
\end{bmatrix}.
\]

b) Factor $B$ into $QAQ^T$ with $Q^{-1} = Q^T$. Draw a graph of the energy function $f(u_1, u_2, u_3) = \frac{1}{2}u^TBu$. This is a surface in 4-dimensional $u_1, u_2, u_3, f$ space so your graph may not be perfect—OK to describe it in 1 sentence.

c) What differential equation with what boundary conditions on $y(x)$ at $x = 0$ and 1 is the continuous analog of $By = \lambda y$? What are the eigenfunctions $y(x)$ and eigenvalues $\lambda$ in this differential equation? At which $x$’s would you sample the first three eigenfunctions to get the three eigenvectors in part (a)?
Solution.

a) \( y_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \) has \( By_1 = 0 \) so \( \lambda_1 = 0 \) (check: trace =4)

The given vector \( y_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \) has \( By_2 = y_2 \) so \( \lambda_2 = 1 \)

\( y_3 \) is orthogonal to \( y_1, y_2 \) \( y_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \) with \( \lambda_3 = 3 \)

b) The orthonormal eigenvectors are the columns of \( Q \) (orthonormal gives \( Q^T Q = I \)).

Then \( B = QAQ^T \) with \( Q = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ \sqrt{3} & \sqrt{2} & \sqrt{6} \end{bmatrix} \) \( \Lambda = \begin{bmatrix} 0 & & 1 \\ & 1 & \\ & & 3 \end{bmatrix} \)

The graph of \( f = \frac{1}{2} u^T B u \) has a valley along the line of eigenvectors \( u = (c, c, c) \). The surface goes up the orthogonal directions.
c) $B$ is free-free so the equation is $-y'' = \lambda y$ with $y' = 0$ at $x = 0, 1$.

$$y_k = \cos k\pi x, \quad k = 0, 1, 2, \ldots$$

Sample at the points $x = (\frac{1}{6}, \frac{3}{6}, \frac{5}{6})$ to get (a multiple of) the discrete eigenvectors $y_1, y_2, y_3$. 

3) **(28 pts.)** The fixed-fixed figure shows \( n = 2 \) masses and \( m = 4 \) springs. Displacements \( u_1, u_2 \).

\[
\begin{array}{c}
  \text{c}_1 \\
  m_1 \\
  \text{c}_2 \\
  m_2 \\
  \text{c}_3 \\
  \text{c}_4 \\
\end{array}
\]

a) Write down the stretching-displacement matrix \( A \) in \( e = Au \).

b) What is the stiffness matrix \( K = A^TCA \) for this system?

c) **Theory question about any** \( A^TCA \). \( C \) is symmetric positive definite. What condition on \( A \) assures that \( u^T A^TCAu > 0 \) for every vector \( u \neq 0 \)? Explain why this is greater than zero and where you use your condition on \( A \).
Solution.

a)

\[
\begin{bmatrix}
e_1 \\
e_2 \\
e_3 \\
e_4 \\
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
-1 & 1 \\
0 & 1 \\
0 & -1 \\
\end{bmatrix}\begin{bmatrix}
u_1 \\
u_2 \\
u_3 \\
u_4 \\
\end{bmatrix} = Au
\]

b)

\[
A^TCA = \begin{bmatrix}
1 & -1 & 0 & 0 \\
0 & 1 & 1 & -1 \\
\end{bmatrix}\begin{bmatrix}
c_1 \\
c_2 \\
c_3 \\
c_4 \\
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
-1 & 1 \\
0 & 1 \\
0 & -1 \\
\end{bmatrix}\begin{bmatrix}
c_1 + c_2 & -c_2 \\
-c_2 & c_2 + c_3 + c_4 \\
\end{bmatrix}
\]

c) Write \( u^T A^T C A u = (A u)^T C (A u) = e^T C e \)

Since \( C \) is positive definite, this is positive unless \( e = 0 \).

**Condition on \( A \): Independent columns.**

Then \( e = Au \) is zero only if \( u = 0 \).

So \( A^T C A \) is positive definite.