Problem 1 (33 points)

This question is about a fixed-free hanging bar (made of 2 materials) with a point load at \( x = \frac{3}{4} \):

\[
- \frac{d}{dx} \left( c(x) \frac{du}{dx} \right) = \delta \left( x - \frac{3}{4} \right)
\]

\[
u(0) = 0
\]

\[
w(1) = 0
\]

Suppose that

\[
c(x) = \begin{cases} 
1, & x < \frac{1}{2} \\
4, & x > \frac{1}{2}
\end{cases}
\]

a) Which of \( u \), \( \frac{du}{dx} \), and \( w = c \frac{du}{dx} \) have jumps at (i) \( x = \frac{1}{2} \) and (ii) \( x = \frac{3}{4} \)?
b) Solve for $w(x)$ and draw its graph from $x = 0$ to $x = 1$.

c) Solve for $u(x)$ and draw its graph from $x = 0$ to $x = 1$. 
Problem 2 (34 points)

a)  
   (i) Find the real part $u(x,y)$ and the imaginary part $s(x,y)$ of
   \[ f(z) = \frac{1}{z} = \frac{1}{x + iy} \]

   (ii) Also find $u(r, \theta)$ and $s(r, \theta)$ for the same function expressed in polar coordinates:
   \[ f(z) = \frac{1}{z} = \frac{1}{re^{i\theta}} \]

b) Draw the equipotential curve $u(x,y) = \frac{1}{2}$ and the streamline $s(x,y) = \frac{1}{2}$. (I suggest to use x-y coordinates and "clear out" denominators.) What shapes are these two curves?

c) What can you say about $u(x,y)$ (what condition does it satisfy) along the line $s = \frac{1}{2}$?
Problem 3 (33 points)
a). Suppose that the Laplacian of $F(x,y)$ is zero:
\[ \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 0. \]
Show that $u = \frac{\partial F}{\partial y}$ and $s = \frac{\partial F}{\partial x}$ satisfy the Cauchy-Riemann equations.

b). Which of these vector fields are gradients of some function $u(x,y)$ and what is that function? Does $u(x,y)$ solve Laplace’s equation $\text{div} (\text{grad} u) = 0$?
(i) $v(x,y) = (x^2,y^2)$

(ii) $v(x,y) = (y^2,x^2)$

(iii) $v(x,y) = (x+y,x-y)$

c) (i) Find the solution to Laplace’s equation inside the unit circle $r^2 = x^2 + y^2 = 1$ if the boundary condition on the circle is $u = u_0(\theta) = \frac{1}{2} + \cos \theta + \cos 2\theta$. (OK to use polar coordinates.)
(ii) Find the numerical value of the solution $u$ at the center and at the point $x = \frac{1}{2}, y = 0$. 