1. (25 points)

(a) The $2\pi$-periodic function $F(x)$ equals 1 for $0 \leq x < \pi$ and equals 0 for $\pi \leq x < 2\pi$. Find its Fourier coefficients $c_k$ using complex exponentials:

$$F(x) = \sum_{-\infty}^{\infty} c_k e^{ikx}$$

Write out the terms for $k = -1, 0, 1$. What is the decay rate of the $c_k$ as $k \to \infty$? How do you see this from the function $F(x)$?

(b) The energy equality connects $\int |F(x)|^2 \, dx$ with $\sum |c_k|^2$. What is this equation for our particular $F(x)$? Find a formula for $\pi$.

(c) What is the derivative of this $F(x)$? Draw the graph of $dF/dx$ !! What is the complex Fourier series for $dF/dx$? What is the decay rate of the coefficients? WHY?
2. (25 points) I am looking for the 7th degree polynomial $p(z) = c_0 + c_1 z + \cdots + c_7 z^7$ that has values 1, 0, 1, 0, 1, 0, 1, 0 at the 8 points $z = 1, z = w, \ldots, z = w^7$. These points are the 8th roots of 1 with $w = e^{2\pi i/8}$ and $w^2 = e^{2\pi i/4} = i$.

(a) We have 8 equations for the 8 $c$’s. The zeroth equation is $p(z) = 1$ at $z = 1$: $c_0 + c_1 + \cdots + c_7 = 1$

What are the next two equations (at $z = w$ and $z = w^2$)? If you put all 8 equations in matrix form $A c = b$, describe the matrix $A$ and the vector $b$.

(b) By knowing the inverse matrix, you can solve those equations. Write down $c = A^{-1} b$. What is that inverse matrix?

(c) Now multiply to find the 8 components of $c$. What is the polynomial $p(z)$? Please check that it has the right values 1, 0, 1, 0, 1, 0, 1, 0 at $z = 1, w, \cdots, w^7$. 

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3. (25 points)

(a) Find the Fourier integral transform $\hat{f}(k)$ of this function $f(x)$:

$$f(x) = 0 \text{ for } x < 0, \ f(x) = e^{-ax} \text{ for } x \geq 0$$

(b) Take the Fourier transform of each term in the differential equation:

$$\frac{du}{dx} + au(x) = \delta(x) \quad -\infty < x < \infty$$

Now find $\hat{u}(k)$. Now find $u(x)$.

(c) Check that your $u(x)$ does solve the differential equation at $x < 0$ and $x = 0$ and $x > 0$. If the right side of the equation changes from $\delta(x)$ to $\delta(x - 1)$, find the new solution $U(x)$:

$$\frac{dU}{dx} + aU(x) = \delta(x - 1)$$
4. (25 points)

(a) These matrix-vector multiplications $Cx$ and $Cy$ are the cyclic convolution of which vectors?

$$Cx = \begin{bmatrix} 5 & -1 & -1 & -1 \\ -1 & 5 & -1 & -1 \\ -1 & -1 & 5 & -1 \\ -1 & -1 & -1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

and

$$Cy = \begin{bmatrix} 5 & -1 & -1 & -1 \\ -1 & 5 & -1 & -1 \\ -1 & -1 & 5 & -1 \\ -1 & -1 & -1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \end{bmatrix}$$

Take the Discrete Fourier Transform of all three vectors $c, x, y$. Call those transforms $\hat{c}, \hat{x}, \hat{y}$.

(b) Convert those two cyclic convolutions $Cx$ and $Cy$ into component-by-component multiplications of the transforms. The answer uses numbers.

(c) Apparently this $y = (1, -1, 1, -1)$ is an eigenvector with $\lambda = 6$. Multiply any circulant matrix $C$ times $y$ to find the eigenvalue:

$$Cy = \begin{bmatrix} c_0 & c_3 & c_2 & c_1 \\ c_1 & c_0 & c_3 & c_2 \\ c_2 & c_1 & c_0 & c_3 \\ c_3 & c_2 & c_1 & c_0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \lambda y.$$  

How is $\lambda$ connected to the transform $\hat{c}$ of $c = (c_0, c_1, c_2, c_3)$?