1) (40 pts.) This question is about $2\pi$-periodic functions.

(a) Suppose $f(x) = \sum c_k e^{ikx}$ and $g(x) = \sum d_t e^{itx}$. Substitute for $f$ and $g$ and integrate to find the coefficients $q_n$ in this convolution:

$$h(x) = \int_0^{2\pi} f(t) g(x-t) dt = \int_0^{2\pi} f(x-t) g(t) dt = \sum q_n e^{inx}.$$

(b) Compute the coefficients $c_k$ for the function

$$f(x) = \begin{cases} 1 & \text{for } 0 \leq x \leq 1 \\ 0 & \text{for } 1 \leq x \leq 2\pi \end{cases}$$

What is the decay rate of these $c_k$? What is $\sum |c_k|^2$?

(c) Keep that $f(x)$ in parts (c)–(d). If $g(x)$ also has a jump, will the convolution $h(x)$ have a jump? Compare the decay rates of the $d$’s and $q$’s to find the behavior of $h(x)$: delta function, jump, corner, or what?

(d) Find the derivative $dh/dx$ at $x = 0$ in terms of two values of $g(x)$.

(You could take the $x$ derivative in the convolution integral.)
2) (30 pts.)  (a) We want to compute the cyclic convolution of \( f = (1, 0, 1, 0) \) and \( g = (1, 0, -1, 0) \) in two ways. First compute \( f \ast_C g \) directly—either the formula at the end of p. 294 or from \( 1 + w^2 \) and \( 1 - w^2 \).

(b) Now compute the discrete transforms \( c \) (from \( f \)) and \( d \) (from \( g \)). Then use the convolution rule to find \( f \ast_C g \).

(c) I notice that the usual dot product \( \overline{f}^T g \) is zero. Maybe also \( \overline{c}^T d \) is zero.

Question for any \( c \) and \( d \):

If \( \overline{c}^T d = 0 \) deduce that \( \overline{f}^T g = 0 \).
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3) \textbf{(30 pts.)} This question uses the Fourier integral to study

\[
-d^2u\over dx^2 + u(x) = \begin{cases} 1 & \text{for } -1 \leq x \leq 1 \\ 0 & \text{for } |x| > 1 \end{cases}
\]

(a) Take Fourier transforms of both sides to find a formula for \( \hat{u}(k) \).

(b) What is the decay rate of this \( \hat{u} \)? At what points \( x \) is the solution \( u(x) \) not totally smooth? Describe \( u(x) \) at those points: delta, jump in \( u(x) \), jump in \( du/dx \), jump in \( d^2u/dx^2 \), or what?

(c) We know that the Green’s function for this equation (when the right side is \( \delta(x) \)) is

\[
G(x) = \frac{1}{2}e^{-|x|} = \begin{cases} \frac{1}{2}e^{-x} & \text{for } x \geq 0 \\ \frac{1}{2}e^x & \text{for } x \leq 0 \end{cases}
\]

Find the solution \( u(x) \) at the particular point \( x = 2 \).