1) (30 pts.) (a) Solve this cyclic convolution equation for the vector $d$. (I would transform convolution to multiplication.) Notice that $c = (5, 0, 0, 0) - (1, 1, 1, 1)$. The equation is like deconvolution.

$$c \ast d = (4, -1, -1, -1) \ast (d_0, d_1, d_2, d_3) = (1, 0, 0, 0).$$

(b) Why is there no solution $d$ if I change $c$ to $C' = (3, -1, -1, -1)$? Try it. Can you find a nonzero $D$ so that $C \ast D = (0, 0, 0, 0)$?
2) (36 pts.)

(a) If \( f(x) = e^{-x} \) for \( 0 \leq x \leq 2\pi \), extended periodically, find its (complex) Fourier coefficients \( c_k \).

(b) What is the decay rate of those \( c_k \) and how could you see the decay rate from the function \( f(x) \)?

(c) Compute \( \sum_{-\infty}^{\infty} |c_k|^2 \) for those \( c \)'s as an ordinary number. [1 point question: How in the world could you find \( \sum_{-\infty}^{\infty} |c_k|^4 \)? Don’t try!]

(d) Solve this periodic differential equation to find \( u(x) \):

\[
u'(x) + u(x) = \delta(x) + \delta(x+2\pi) + \delta(x-2\pi) + \cdots \text{ train of deltas}
\]
3) (34 pts.) Suppose $f(x)$ is a half-hat function $(-\infty < x < \infty)$.

$$f(x) = \begin{cases} 
1 - x & \text{for } 0 \leq x \leq 1 \\
0 & \text{for all other } x
\end{cases}$$

(a) Draw a graph of $f(x)$ on the whole line $-\infty < x < \infty$ and ALSO a graph of its derivative $g(x) = df/dx$.

(b) What is the transform (Fourier integral) $\widehat{g}(k)$ of $df/dx$?

(c) What is the transform $\widehat{f}(k)$ of $f(x)$? Does it have the decay rate you expect? What is $\widehat{f}(0)$?

(d) Christmas present: Is the convolution $f(x) \ast f(x)$ of the half-hat with itself equal to the usual full hat $H(x)$? (Yes or no answer, 4 points).

THANK YOU FOR TAKING 18.085! 18.086 will be good small projects in scientific computing.