18.085/18.0851 Computational Science and Engineering I **Summer 2020**

Homework 5

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5.1 Matching Initial Conditions

We showed in class that for a damped harmonic oscillator of damping constant ν, mass m, and spring constant c, the solution would be,

$$u(t) = A_1 e^{b_1 t} + A_2 e^{b_2 t} \tag{5.1}$$

where $b_1 = \frac{-\nu+\sqrt{\nu^2-4mc}}{2m}$ and $b_2 = \frac{-\nu-\sqrt{\nu^2-4mc}}{2m}$ Suppose that we have the initial conditions $u(0) = 0$ and $u'(0) = 1$.

- Solve for the constants A_1 and A_2, for both the underdamped and overdamped system.
- Show that (5.51) from the Week 5 Lecture Notes renders the same result

5.2 Nondimensionalized equation

We recall in class that in a damped harmonic oscillator, if we let $\gamma = \frac{\nu}{2m}$ and $\omega_0 = \sqrt{\frac{c}{m}}$, then the equation can be written as

$$\frac{d^2q}{dt^2} + 2\gamma\frac{dq}{dt} + \omega_0^2 q(t) = 0 \tag{5.2}$$

where q is some normalized measure of $u(t)$ (the extension of the mass). Conduct the same analysis as we did in class and figure out the relation between γ and ω_0 that determines whether a system is overdamped or underdamped.

5.3 Laplace Transform

Solve the following differential equation using the Laplace's transform

$$u'' + u = 1 \tag{5.3}$$

$$u(0) = u'(0) = 0 \tag{5.4}$$

5.4 Fourier Series: Sine and Cosine basis

Using sines and cosines, find the Fourier series representation of the following function

$$f(x) = x, -\pi \leq x \leq \pi \tag{5.5}$$

5.5 Orthogonality Relations

Prove that over the interval $[-L, L]$, the following relations hold true

- $\sin\frac{n\pi}{L}x, \sin\frac{k\pi}{L}x \quad = L\delta_{kn}$
- $\cos\frac{k\pi}{L}x, \cos\frac{n\pi}{L}x \quad = L\delta_{kn}$
- $\sin\frac{n\pi}{L}x, \cos\frac{n\pi}{L}x \quad = 0$

Recall that

- $\langle f, g\rangle = \int_{-L}^{L} fgdx$
- $\delta_{nk} = 0$ if $n \neq k$ and $\delta_{nk} = 1$ if $n = k$

Now if we have a function $f(x)$ defined over $[-L, L]$, we can write the Fourier series as

$$f(x) = \sum_{n=0}^{\infty} a_n \cos\frac{n\pi}{L}x + \sum_{n=0}^{\infty} b_n \sin\frac{n\pi}{L}x \tag{5.6}$$

Write down the formula for a_n and b_n Hint: if you are stuck with the integrals, try the following trig identity

$$\cos(a)\cos(b) = \frac{1}{2}(\cos(a-b) + \cos(a+b)) \tag{5.7}$$

$$\cos(a)\sin(b) = \frac{1}{2}(\sin(b+a) + \sin(b-a)) \tag{5.8}$$

$$\sin(a)\sin(b) = \frac{1}{2}(\cos(a-b) - \cos(a+b)) \tag{5.9}$$