18.085 Pset #5 Solutions

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Question 1.

At $t = 0$ we have $u(0) = 0 = A_1 + A_2$ so $A_1 = -A_2$. The derivative is $u'(t) = A_1 b_1 e^{b_1 t} + A_2 b_2 e^{b_2 t}$ so at $t = 0$ we get: $u'(0) = 1 = A_1 b_1 + A_2 b_2$. Then we get:

$$A_1 b_1 = A_1 b_2 + 1 \implies A_1 = \frac{1}{b_1 - b_2}$$

Then the given b_1 and b_2 can be substituted in to determine A_1 and A_2:

$$A_1 = \frac{m}{\sqrt{\nu^2 - 4mc}} \qquad A_2 = -\frac{m}{\sqrt{\nu^2 - 4mc}}$$

Question 2.

Let us guess solution $q(t) = Ae^{bt}$. Then we have $q'(t) = Abe^{bt}$ and $q''(t) = Ab^2 e^{bt}$. Substituting these into the diffeq we have:

$$\begin{aligned} Ab^2 e^{bt} + 2\gamma Abe^{bt} + \omega_0^2 Ae^{bt} &= 0 \\ Ae^{bt}(b^2 + 2\gamma b + \omega_0^2) &= 0 \\ \implies b = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2} \end{aligned}$$

- System is overdamped when $\gamma^2 - \omega_0^2 > 0$
- System is underdamped when $\gamma^2 - \omega_0^2 < 0$
- System is critically damped when $\gamma^2 - \omega_0^2 = 0$

These are determined by checking if b is complex or real. In the case $\gamma^2 - \omega_0^2 < 0$ then b is complex and $e^{bt} = e^{-\gamma} e^{i\sqrt{\omega_0^2 - \gamma^2}}$ has an oscillatory component.

Question 3.

Taking the Laplace transform of both sides we see:

$$\begin{aligned} u'' + u &= 1 \\ \mathcal{L}\{u'' + u\}(s) &= \mathcal{L}\{1\}(s) \\ (s^2U(s) - su(0) - u'(0)) + U(s) &= \frac{1}{s} \\ s^2U(s) + U(s) &= \frac{1}{s} \\ U(s) &= \frac{1}{s(s^2+1)} \end{aligned}$$

By a partial fractions decomposition we can find this is equivalent to

$$U(s) = \frac{1}{s} - \frac{s}{s^2+1}$$

Converting back into the time domain:

$$u(t) = 1 - \cos t$$

which satisfies the initial conditons $u(0) = u'(0) = 0$.

Question 4.

The function is $f(x) = x$, $-\pi \le x \le \pi$ and we wish to find constants a_n and b_n such that $f(x) = \sum a_n \cos(nx) + \sum b_n \sin(nx)$. We can immediately see that because $f(x)$ is an odd function on the interval, the a_n are all zero. To find b_n we use:

$$\begin{aligned} b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(kx)dx &= \frac{1}{\pi}\left[-\frac{x\cos(kx)}{k}\right]_{-\pi}^{\pi} + \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\cos(kx)}{k}dx \\ &= \left[-\frac{x\cos(kx)}{k} + \frac{\sin(kx)}{k^2}\right]_{-\pi}^{\pi} \\ &= \frac{1}{\pi}\left(\frac{\sin(k\pi) - k\pi\cos(k\pi)}{k^2} - \frac{\sin(-k\pi) - k\pi\cos(k\pi)}{k^2}\right) \\ &= \frac{2}{\pi k^2}\left(\sin(k\pi) - k\pi\cos(kx)\right) = 2(-1)^{k+1}/k \end{aligned}$$

So we have:

$$f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}\sin(nx)}{n}$$

Question 5.

First relation when $n \neq m$:

$$\begin{aligned}
\left\langle \sin\frac{n\pi}{L}x, \sin\frac{m\pi}{L}x \right\rangle &= \int_{-L}^{L} \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right) dx \\
&= \int_{-L}^{L} \frac{1}{2}\left(\cos\left(\frac{n\pi}{L}x - \frac{m\pi}{L}x\right) - \cos\left(\frac{n\pi}{L}x + \frac{m\pi}{L}x\right)\right) dx \\
&= \frac{1}{2}\left[\frac{L}{\pi(n-m)}\sin\left(\frac{\pi(n-m)}{L}x\right)\right]_{-L}^{L} - \frac{1}{2}\Big[\cdots\Big]_{-L}^{L} \\
&= \frac{-L}{\pi(n^2-m^2)}\Big((n-k)\sin(\pi(n+k)) - (n+k)\sin(\pi(n-k))\Big) \\
&= 0
\end{aligned}$$

When $n = m$ we have the case:

$$\int_{-L}^{L} \sin^2\left(\frac{n\pi}{L}x\right) dx = \frac{1}{2}\int_{-L}^{L}\left(1 - \cos\left(\frac{2\pi n}{L}x\right)\right) dx = L$$

Therefore:

$$\left\langle \sin\frac{n\pi}{L}x, \sin\frac{m\pi}{L}x \right\rangle = L\delta_{nm}$$

The other relations can be checked in much the same way using the given trig idenities. To determine a_n and b_n, take the inner product of $f(x)$ with $\sin\frac{n\pi}{L}x$ and $\cos\frac{n\pi}{L}$ by expanding $f(x)$ into its Fourier series:

$$\begin{aligned}
\left\langle f(x), \sin\frac{n\pi}{L}x \right\rangle &= \left\langle \sum_{k=0}^{\infty} a_k \cos\frac{k\pi}{L}x + \sum_{k=1}^{\infty} b_k \sin\frac{k\pi}{L}x, \sin\frac{n\pi}{L}x \right\rangle \\
&= \left\langle b_n \sin\frac{n\pi}{L}x, \sin\frac{n\pi}{L} \right\rangle = b_n L
\end{aligned}$$

Where all the other terms of the Fourier series vanish from the inner product because of orthogonality. Then:

$$b_n = \frac{1}{L}\left\langle f(x), \sin\frac{n\pi}{L}x \right\rangle = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x\, dx$$

And similarly:

$$\left\langle f(x), \cos\frac{n\pi}{L}x \right\rangle = \left\langle \sum_{k=0}^{\infty} a_k \cos\frac{k\pi}{L}x + \sum_{k=1}^{\infty} b_k \sin\frac{k\pi}{L}x, \cos\frac{n\pi}{L}x \right\rangle$$
$$= \left\langle a_n \cos\frac{n\pi}{L}x, \cos\frac{n\pi}{L} \right\rangle = a_n L$$
$$\Longrightarrow \; a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\,dx$$

a_0 is a special case where the inner product is just $\langle a_0 \cdot 1, 1\rangle = a_0 2L$, so

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$