18.085 Pset #6 Solutions

August Trollbäck

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Question 1.

In the Fourier series $f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$ we find c_k using:

$$\begin{aligned}
c_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{-ikx}dx \\
&= \frac{1}{2\pi}\left[\frac{x}{-ik}e^{-ikx}\right]_{-\pi}^{\pi} - \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{-ik}e^{-ikx}dx \\
&= \frac{1}{2\pi}\left[\frac{x}{-ik}e^{-ikx}\right]_{-\pi}^{\pi} - \frac{1}{2\pi}\left[-\frac{1}{k^2}e^{-ikx}\right]_{-\pi}^{\pi} \\
&= \frac{1}{2\pi}\left[\left(\frac{ix}{k}+\frac{1}{k^2}\right)e^{-ikx}\right]_{-\pi}^{\pi} = \frac{i(-1)^k}{k}
\end{aligned}$$

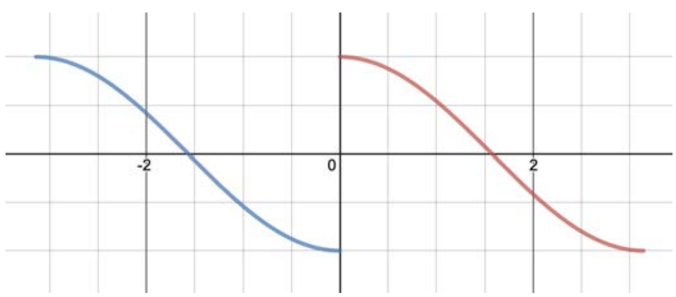
For c_0 we have $c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,dx = 0.$

From the last homework we saw that $x = \sum_{k=1}^{\infty} -\frac{2(-1)^k}{k}\sin(kx)$. We can check that this matches our complex Fourier series:

$$
\begin{aligned}
\sum_{k=-\infty}^{\infty} c_k e^{ikx} &= c_0 + \sum_{k=1}^{\infty}\left(c_k e^{ikx} + c_{-k}e^{-ikx}\right) \\
&= \sum_{k=1}^{\infty}\left(\frac{i(-1)^k}{k}\left(\cos(kx) + i\sin(kx)\right) - \frac{i(-1)^{-k}}{k}\left(\cos(-kx) + i\sin(-kx)\right)\right) \\
&= \sum_{k=1}^{\infty}\left(\frac{i(-1)^k}{k}\left(\cancel{\cos(kx)} + i\sin(kx)\right) - \frac{i(-1)^k}{k}\left(\cancel{\cos(kx)} - i\sin(kx)\right)\right) \\
&= \sum_{k=1}^{\infty}\left(\frac{2i(-1)^k}{k} i\sin(kx)\right) = \sum_{k=1}^{\infty} -\frac{2(-1)^k}{k}\sin(kx) \quad \checkmark
\end{aligned}
$$

Question 2.

- The odd extension of $f(x) = \cos x$ looks like:



We know the a_n terms are zero, and we find the b_n with:

$$\begin{aligned}
b_k &= \frac{1}{L}\int_{-L}^{L} g(x)\sin\frac{k\pi}{L}x\,dx \\
&= \frac{2}{L}\int_{0}^{L} \cos(x)\sin\frac{k\pi}{L}x\,dx \\
&= \frac{1}{L}\int_{0}^{L} \left(\sin\left(\frac{k\pi}{L}x + x\right) + \sin\left(\frac{k\pi}{L}x - x\right)\right)dx \\
&= \frac{1}{L}\left[\frac{-L}{k\pi + L}\cos\left(\frac{k\pi + L}{L}x\right) - \frac{L}{k\pi - L}\cos\left(\frac{k\pi - L}{L}x\right)\right]_0^L \\
&= \frac{2k\pi - (k\pi - L)\cos(k\pi + L) - (k\pi + L)\cos(k\pi - L)}{k^2\pi^2 - L^2}
\end{aligned}$$

- The even extension of $f(x) = \cos x$ is simply $g(x) = \cos x$ on $[-L, L]$. We know the b_n terms are zero, and we find the a_n with:

$$\begin{aligned}
a_k &= \frac{1}{L}\int_{-L}^{L} g(x)\cos\frac{k\pi}{L}x\,dx \\
&= \frac{2}{L}\int_{0}^{L} \cos(x)\cos\frac{k\pi}{L}x\,dx \\
&= \frac{1}{L}\int_{0}^{L} \left(\cos\left(x - \frac{k\pi}{L}x\right) + \cos\left(x + \frac{k\pi}{L}x\right)\right)dx \\
&= \frac{1}{L}\left[\frac{L}{L - k\pi}\sin\left(x - \frac{k\pi}{L}x\right) + \frac{L}{L + k\pi}\sin\left(x + \frac{k\pi}{L}x\right)\right]_0^L \\
&= \frac{1}{L - k\pi}\sin(L - k\pi) + \frac{1}{L + k\pi}\sin(L + k\pi)
\end{aligned}$$

Question 3.

Note that this is exactly as the same as the example in section 7.2.1 of the lecture notes, with the exception that x and y have swapped places and the case $L = 1$. Following the steps shown there, the result would be:

$$u(x, y) = \sum_{n=1}^{\infty} \frac{2}{n\pi(e^{n\pi} - e^{-n\pi})}(-1)^n(e^{n\pi x} - e^{-n\pi x})\sin(n\pi y)$$

Question 4.

Our general solution is:

$$u(r,\theta) = A_0(C_0 \ln r + D_0) + \sum_{m=1}^{\infty} (A_m \cos m\theta + B_m \sin m\theta)(C_m r^m + D_m r^{-m})$$

If we assume $C_0 = 0$ and $D_k = 0$ so that $u(r,\theta)$ is bounded on the interior domain, and reinterpret A_k and B_k to include C_k we have a simplified general solution:

$$u(r,\theta) = A_0 + \sum_{m=1}^{\infty} (A_m \cos m\theta + B_m \sin m\theta) r^m$$

At $r = 1$ we have

$$u(1,\theta) = \cos 10\theta = A_0 + \sum_{m=1}^{\infty} (A_m \cos m\theta + B_m \sin m\theta)$$

So we can the determine the proper constants are $A_k = \delta_{10,k}$ (1 for $m = 10$ and 0 otherwise) and $B_k = 0$:

$$u(r,\theta) = \cos(10\theta) r^{10}$$