**18.085/18.0851 Comp. Science. Engineering I** **Name:** ____________
**Summer 2020**
**Final Examination**
**8/10/2020**
**Time Limit: Unlimited** ____________

---

This exam contains 11 pages (including this cover page) and 7 questions plus 1 bonus questions.

Total of points is 63.

You are allowed all your written notes for the course. There will be no use of electronic devices except for looking up specific MATLAB commands online

Grade Table (for instructor use only)

| Question | Points | Score |
|---|---|---|
| 1 | 4 | |
| 2 | 4 | |
| 3 | 5 | |
| 4 | 5 | |
| 5 | 10 | |
| 6 | 25 | |
| 7 | 10 | |
| Total: | 63 | |

---

1. (4 points) Which of the following are the appropriate Fourier basis functions over the interval $(-L, L)$? Here $n$ is an integer. Select all that applies
   A. $\sin(nx/L), \cos(nx/L)$. **INCORRECT**
   B. $\sin(n\pi x/L), \cos(n\pi x/L)$. **CORRECT**
   C. $e^{\pm inx/L}$. **INCORRECT**
   D. $e^{\pm in\pi x/L}$. **CORRECT**

2. (4 points) Which of the following are suitable Ansatz to the damped harmonic oscillator,

$$m\frac{d^2u}{dt^2} + \nu\frac{du}{dt} + cu = 0 \tag{1}$$

   where $u$ describes the displacement from equilibrium a ball of mass $m$ connected to a spring with Hooke's constant $c$ and experiencing a friction with coefficient $\nu$. Circle all that applies

A. $u(t) = \alpha \cos(bt) + \beta \sin(bt)$, with $\alpha, \beta, b \in \mathbb{R}$ to be solved. **INCORRECT**

B. $u(t) = A \exp(bt)$, with $A, b \in \mathbb{C}$ to be solved. **CORRECT**

C. $u(t) = A \exp(ibt^2)$, with $A, b \in \mathbb{C}$ to be solved. **INCORRECT**

D. $u(t) = A \exp(ibt)$, with $A, b \in \mathbb{C}$ to be solved. **CORRECT**

3. (5 points) Circle all true statements

A. The solution to the heat equation in $\mathbb{R}$ can be discontinuous, depending on the initial condition. **INCORRECT**

B. The non-smoothness of the boundary condition of the Laplace's equation can chip away the convergence rate of the finite difference approximation. **CORRECT**

C. In an under-damped system, the damping effect is dominated by the inertia and/or the stiffness of the spring. **CORRECT**

D. There should not be any terms involving $r^m$ for $m > 0$ in the solution to the **interior** domain of the Laplace's equation. Here $r$ is the radial coordinate in the polar system. **INCORRECT**

E. The finite element method via mesh generation is less flexible in adapting a complex geometry of the domain compared with the finite difference method. **INCORRECT**

4. (5 points) Suppose I perform the FFT on a 128 data points, originally generated by the signal $\sin(2\pi * 3t)$, where $t \in [0.5362, 3.5345]$. Then when I plot $k$ vs. the magnitudes of discrete Fourier transform coefficients, $\hat{u}_k^d$, for $k = 0, ...127$, for what value(s) of $k$ should I anticipate the plot to peak? Circle all that applies

A. $k = 0$. **INCORRECT**

B. $k = 3$. **CORRECT**

C. $k = 124$. **INCORRECT**

D. $k = 125$. **CORRECT**

E. $k = 127$. **INCORRECT**

| $f(t)$ | $\mathscr{L}\{f(t)\} = F(s)$ | *Region of convergence* |
|---|---|---|
| $c$, $c$ a constant | $\frac{c}{s}$ | $\mathrm{Re}(s) > 0$ |
| $t$ | $\frac{1}{s^2}$ | $\mathrm{Re}(s) > 0$ |
| $t^n$, $n$ a positive integer | $\frac{n!}{s^{n+1}}$ | $\mathrm{Re}(s) > 0$ |
| $e^{kt}$, $k$ a constant | $\frac{1}{s-k}$ | $\mathrm{Re}(s) > \mathrm{Re}(k)$ |
| $\sin at$, $a$ a real constant | $\frac{a}{s^2+a^2}$ | $\mathrm{Re}(s) > 0$ |
| $\cos at$, $a$ a real constant | $\frac{s}{s^2+a^2}$ | $\mathrm{Re}(s) > 0$ |
| $e^{-kt}\sin at$, $k$ and $a$ real constants | $\frac{a}{(s+k)^2+a^2}$ | $\mathrm{Re}(s) > -k$ |
| $e^{-kt}\cos at$, $k$ and $a$ real constants | $\frac{s+k}{(s+k)^2+a^2}$ | $\mathrm{Re}(s) > -k$ |

5. (10 points) Solve the following differential equation for $t > 0$ using Laplace transform

$$\begin{cases} \frac{d^2u}{dt^2} - \frac{du}{dt} = 0 \\ u(0) = 0 \\ u'(0) = 1 \end{cases} \tag{2}$$

*Solution:*

We define $U(s)$ to be the Laplace transform: $U(s) = \int_0^\infty u(t)e^{-st}dt$. We can then apply the Laplace transform on both sides of the equation:

$$\int_0^\infty \frac{d^2u}{dt^2}e^{-st}dt - \int_0^\infty \frac{du}{dt}e^{-st}dt = 0 \tag{3}$$

Integration by parts on the first term gives:

$$\begin{aligned} \int_0^\infty \frac{d^2u}{dt^2}e^{-st}dt &= \frac{du}{dt}e^{-st}|_{t=0}^\infty - \int_0^\infty \frac{du}{dt}(-s)e^{-st}dt && (4) \\ &= -1 + s\int_0^\infty \frac{du}{dt}e^{-st}dt && (5) \\ &= -1 + sue^{-st}|_{t=0}^\infty + s^2\int_0^\infty ue^{-st}dt && (6) \\ &= -1 + s^2U(s) && (7) \end{aligned}$$

while integration by parts on the second term gives

$$\begin{aligned} \int_0^\infty \frac{du}{dt}e^{-st}dt &= ue^{-st}|_{t=0}^\infty + s\int_0^\infty ue^{-st}dt && (8) \\ &= sU(s) && (9) \end{aligned}$$

Hence, the transformed equation looks like

$$-1 + s^2U(s) - sU(s) = 0 \tag{10}$$

$$U(s) = \frac{1}{s(s-1)} \tag{11}$$

$$U(s) = \frac{1}{s-1} - \frac{1}{s} \tag{12}$$

To get $u(t)$, we check the Laplace transform table for the inverse Laplace transform of $1/s$ and $1/(s-1)$ and realize that $u(t) = e^t - 1$

6. (25 points) Suppose we have the Laplace's equation $\Delta u = 0$ INSIDE a square defined by $x = 0$, $x = 1$, $y = 0$, and $y = 1$, subject to the following boundary condition

$$\begin{cases} u(x,0) = 0 \\ u(x,1) = 0 \\ u(0,y) = 0 \\ u(1,y) = \sin(n\pi y) \end{cases} \tag{13}$$

where $n$ is a fixed integer.

a)(10 points) Solve the Laplace equation analytically.
Two hints:

- Hint 1: Recall the orthogonality of $\sin(l\pi x)$ for $l = 1, 2, 3, ....$ By leveraging the orthogonality, you should not have to do any Fourier integrals explicitly
- Hint 2: Be very careful with dummy and non-dummy indices. Here $n$ is a non-dummy index.

*Solution:*

We postulate based on the separation of variable argument that $u = X(x)Y(y)$, so that

$$\Delta u = X''Y + Y''X = 0 \tag{14}$$

$$\frac{X''}{X} + \frac{Y''}{Y} = 0 \tag{15}$$

We let $Y''/Y = -\lambda^2$ due to its periodicity, which means that $X''/X = \lambda^2$. This gives that

$$Y = A\cos(\lambda y) + B\sin(\lambda y) \tag{16}$$

$$X = Ce^{\lambda x} + De^{-\lambda x} \tag{17}$$

Now we shall work out the boundary conditions:

- $u(x, y = 0) = 0$ means that $Y(y = 0) = 0$, which implies that $A = 0$
- $u(x, y = 1) = 0$ means that $Y(y = 1) = 0$, which implies that $\lambda = k\pi$, for $k = 1, 2, 3, ....$
- $u(x = 0, y) = 0$ means that $X(x = 0) = 0$, which implies that $C = D$

Now we are left with

$$u(x,y) = \sum_{k=1}^{\infty} B_k(e^{k\pi x} - e^{-k\pi x})\sin(k\pi y) \tag{18}$$

where the constants $C$ have been absorbed into $B_k$ for each $k = 1, 2, 3....$ Finally, at $x = 1$, $u(x = 1, y) = \sin(n\pi y)$, whence

$$\sin(n\pi y) = \sum_{k=1}^{\infty} B_k(e^{k\pi} - e^{-k\pi}) \sin(k\pi y) \tag{19}$$

Performing the odd extension of $\sin(n\pi y)$, we observe that due to the orthognality of $\sin(k\pi y)$ over $(-1, 1)$, for $k = 1, 2, 3, ...$, the only nonzero $B_k$ is for $k = n$, whence $u(x, y)$ can be written as

$$\sin(n\pi y) = B_n(e^{n\pi} - e^{-n\pi}) \sin(n\pi y) \tag{20}$$

$$B_n = \frac{1}{e^{n\pi} - e^{-n\pi}} \tag{21}$$

Hence, the solution would look like

$$u(x, y) = \frac{e^{n\pi x} - e^{-n\pi x}}{e^{n\pi} - e^{-n\pi}} \sin(n\pi y) \tag{22}$$

b)(10 points) We set up the rectangular grid by dividing up the $x-axis$ and $y-axis$ into $N-by-N$ grids and run the finite difference scheme to obtain the numerical solution at each inner node $(x_i, y_j)$, where $i = 1, ..., N$ and $j = 1, ..., N$. Note that $x_0 = 0$, $x_{N+1} = 1$, $y_0 = 0$ and $y_{N+1} = 1$. Let $l$ be the linearized index (aka. global index in my notes) of all inner nodes. Note that $l = 1, 2, ..., N^2$.

Suppose the MATLAB function "$laplace1(N)$" implements the second-order finite difference scheme on the $N-by-N$ grid and spits out the numerical solution at each inner node $(l = 1, ..., N^2)$ . On the other hand, suppose the MATLAB function $laplace2(N)$" implements the formula in part a) and spits out the analytic solution at the same inner nodes.

Now we want to evaluate the order of convergence, measured by the $L^2-$ error, as $N$ gets large. Below you can find some incomplete code. **You are welcome to look up MATLAB commands online, but please do not open any codes on Stellar**:

i) Fill up the missing line (denoted as "......")
ii) Describe the output of the code. Here is the instruction for your answer:

- Is the output a number, an array, a plot, or something else?
- If it is a number or array, explicitly write it out or describe what each entry is.
- If it is a plot, indicate key features of the plot ( x-axis, y-axis, slope (if it's a line), period (if it's a sine function), etc).

---

```
N_max = 100;

error_L_2 = zeros(N_max, 1);

for N=1:N_max
    u = laplace1(N);
    v = laplace2(N);
    error_val = 0;

    for n=1:N^2
        error_val = ......
    end

    error_val = sqrt(1/N * error_val);
    error_L_2(N) = error_val;
end

N_array = 1:1:N_max;
loglog(N_array, error_L_2)
```

*Solution:*

The missing line should be

```
error_val = error_val + (u(n)-v(n))^2
```

The output is supposed to be a plot with the $N$, the number of grid points, on the x-axis, and the $L_2$ error on the y-axis, both of which are plotted on log scales. The plot should be a line of slope close to $-2$ because the second derivative finite difference approximation is second order

c) (5 points) Suppose now we try to use the same machinery in part a) and the code in part b) for the problem with the following boundary condition:

$$\begin{cases} u(x,0) = 1 \\ u(x,1) = 1 \\ u(0,y) = 1 \\ u(1,y) = \sin(n\pi y) \end{cases} \tag{23}$$

How and why would the output of the code in part b) change? Here's the instruction for your answer:

- If it is a number or array, explicitly write them out or describe how each entry changes.
- If it is a plot, indicate how certain key features of the plot ( x-axis, y-axis, slope (if it's a line), period (if it's a sine function), etc).

*Solution:*

The plot will have a slope that is much more different from $-2$. The absolute value of the slope would be smaller than 2 and/or close to 1. This is due to the discontinuity in the boundary data, which shakes the validity of the finite difference approximation.

7. (10 points) Compute the Fourier transform of the following function (factor of $2\pi$ would be forgiven)

$$f(x) = \begin{cases} 0, & x < 0 \\ e^{-x}, & x \geq 0 \end{cases} \tag{24}$$

We compute that

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \tag{25}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-x} e^{-ikx} dx \tag{26}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{e^{-(1+ik)x}}{-(1+ik)} |_{x=0}^{\infty} \tag{27}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{1+ik} \tag{28}$$

Alternative forms of the same answer will be accepted.

(Bonus) Please summarize your understandings of the research talks on Wednesday, August 12th, 2020. A reasonable summary of each research talk will be worth 2 points, and up to 8 points will be granted. This part is optional and is due by 11:59pm of August 12, 2020.