DYNAMICAL SYSTEMS AND TRAFFIC CONTROL

Invited Guest Lecture in 18.085/18.0851 Computational Science and Engineering I

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Massachusetts Institute of Technology, 12th August 2020

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Dynamical systems

First-order continuous-time homogeneous system of linear ordinary differential equations:

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t) \cdot \mathbf{x}(t)$$

$$\dot{\mathbf{x}}(t) - \mathbf{A}(t) \cdot \mathbf{x}(t) = 0$$

If $\mathbf{A}(t) = \mathbf{A}$ (LTI system) and $\mathbf{A}$ has n linearly independent eigenvectors, the system has the following general solution:

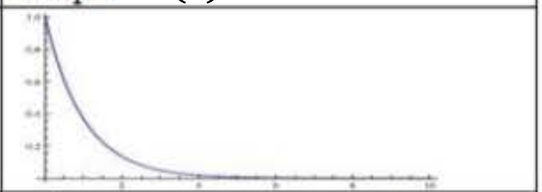
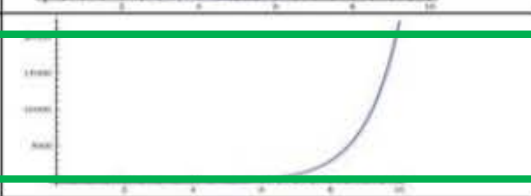
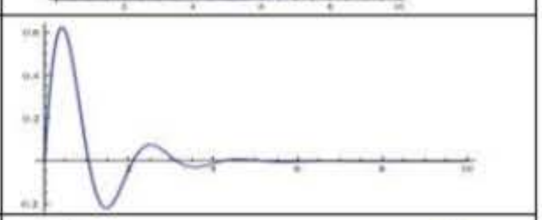
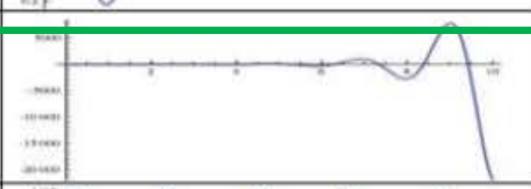
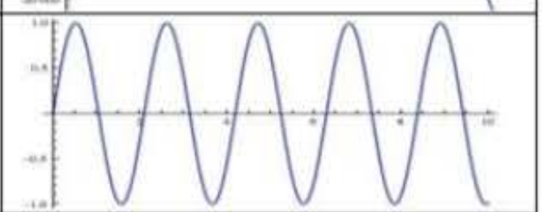
$$\mathbf{x}(t) = c_1 \cdot e^{\lambda_1 t} \cdot \boldsymbol{u_1} + c_2 \cdot e^{\lambda_2 t} \cdot \boldsymbol{u_2} + \cdots + c_n \cdot e^{\lambda_n t} \cdot \boldsymbol{u_n}$$

$\lambda_1, \lambda_2, \dots, \lambda_n$ eigenvalues of $\mathbf{A}$

$\boldsymbol{u_1}, \boldsymbol{u_2}, \dots, \boldsymbol{u_n}$ eigenvectors of $\mathbf{A}$

$c_1, c_2, \dots, c_n$ constants

Stability (one dimensional graph)

Eigenvalues	Graph $\mathbf{x}(t)$
All real and negative	
All real and one or more are positive	
All real eigenvalues are negative and there are imaginary parts	
One or more eigenvlaues have a positive real part and there are imaginary parts	
Real parts of the eigenvlaues are zero and there are imaginary parts	

Asymptotically stable

Asymptotically stable (with oscillations)

Source: https://web2.qatar.cmu.edu/~gdicaro/15382-Spring18/slides/382-S18-5-DynamicalSystems-4.pdf

Traffic Density

Usually indicated by Greek letter ρ.

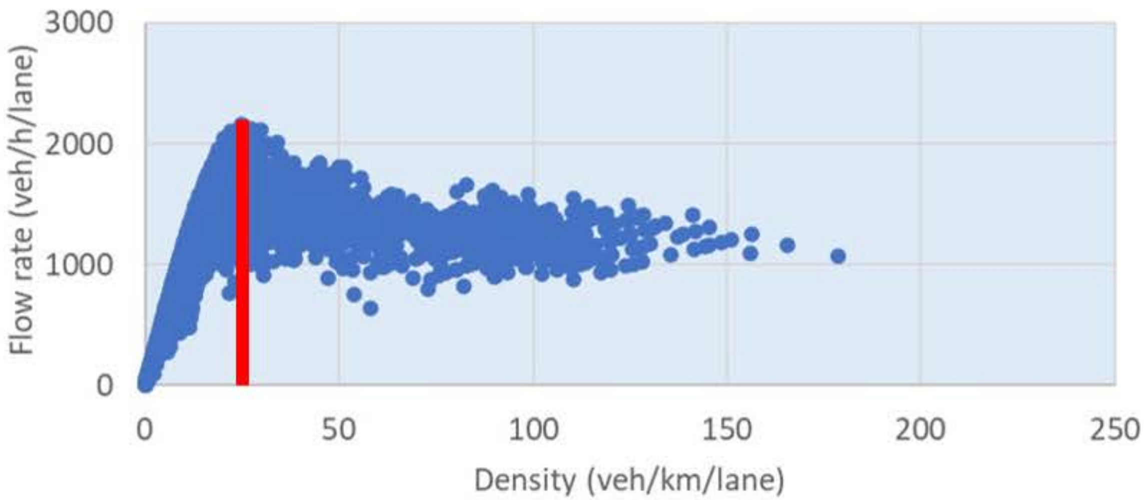
It is the number of vehicles occupying a unit length of roadway.

$$\rho = \frac{q}{v}$$

ρ traffic density [n° of vehicles/km]

q traffic flow [n° of vehicles/h]

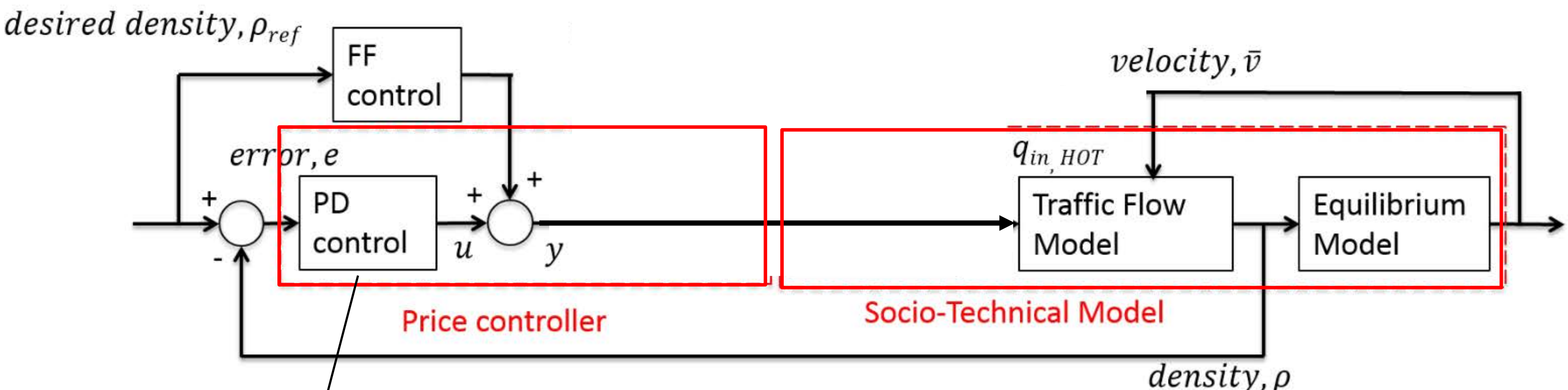
v traffic average speed [km/h]



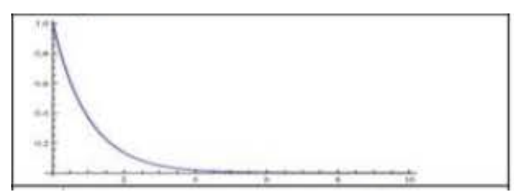
Traffic control

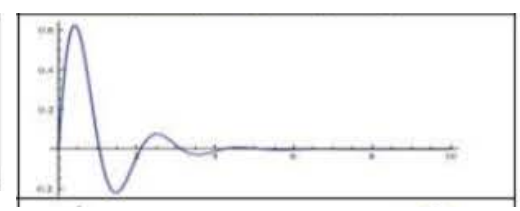
(Phan et al., 2016):

$$\boxed{(e(t) = 0, \dot{e}(t) = 0) \text{ equilibrium point}}$$



$$\boxed{u = K_p \cdot e + K_d \cdot \dot{e}}$$





$$\boxed{\dot{\rho}(t) = a \cdot \rho(t) + b \cdot \rho_{ref}(t)}$$

Traffic control

$$\dot{\rho}(t) = a \cdot \rho(t) + b \cdot \rho_{ref}(t)$$

$$\dot{\rho}(t) - a \cdot \rho(t) = b \cdot \rho_{ref}(t)$$

$$\dot{\mathrm{x}}(t) - a \cdot \mathrm{x}(t) = b \cdot u(t)$$

It is a first-order continuous-time forced linear ordinary differential equation.

In traffic engineering the forcing term is typically constant:

$$\rho_{ref}(t) = \rho_{ref}$$

Traffic control

$$\dot{x}(t) - a \cdot x(t) = b \cdot u(t)$$

The properties of the solution of the forced system are still dictated by the associated homogeneous system:

$$\dot{x}(t) - a \cdot x(t) = 0$$

The eigenvalue(s) of the system are related to the definition of parameters K_p and K_d. If K_p and K_d are chosen so that the eigenvalues of the system are in the portion of the complex plane with negative real part, asymptotic stability of the solution is guaranteed, meaning that, theoretically, density should tend to the desired value asymptotically.

$\dot{\rho}(t) = 0$ and $\rho(t) = \rho_{ref}$ is the equilibrium point of the system we are aiming at.

Traffic control – multiple control sections

If density is controlled only in one section of the facility, we have an ordinary differential equation:

$$\dot{\rho}(t) - a \cdot \rho(t) = b \cdot \rho_{ref}(t)$$

If density is controlled in multiple points of the facility, we have a system of ODEs:

$$\dot{\boldsymbol{\rho}}(t) - \mathbf{A} \cdot \boldsymbol{\rho}(t) = \mathbf{B} \cdot \boldsymbol{\rho}_{\boldsymbol{ref}}(t)$$

Previous considerations regarding stability are still valid.

Digital traffic control

Typically controllers are digital, meaning that they work in discrete-time. If density is controlled only in one section of the facility, we have a recursive equation:

$$\rho(k+1) - a \cdot \rho(k) = b \cdot \rho_{ref}(k)$$

If density is controlled in multiple points of the facility, we have a system of recursive equations:

$$\boldsymbol{\rho}(k+1) - \mathbf{A} \cdot \boldsymbol{\rho}(k) = \mathbf{B} \cdot \boldsymbol{\rho}_{\boldsymbol{ref}}(k)$$

Digital traffic control

For the system

$$\mathbf{x}(k+1) - \mathbf{A} \cdot \mathbf{x}(k) = 0$$

The solution is

$$\mathbf{x}(k) = c_1 \cdot \lambda_1^k \cdot \boldsymbol{u_1} + c_2 \cdot \lambda_2^k \cdot \boldsymbol{u_2} + \cdots + c_n \cdot \lambda_n^k \cdot \boldsymbol{u_n}$$

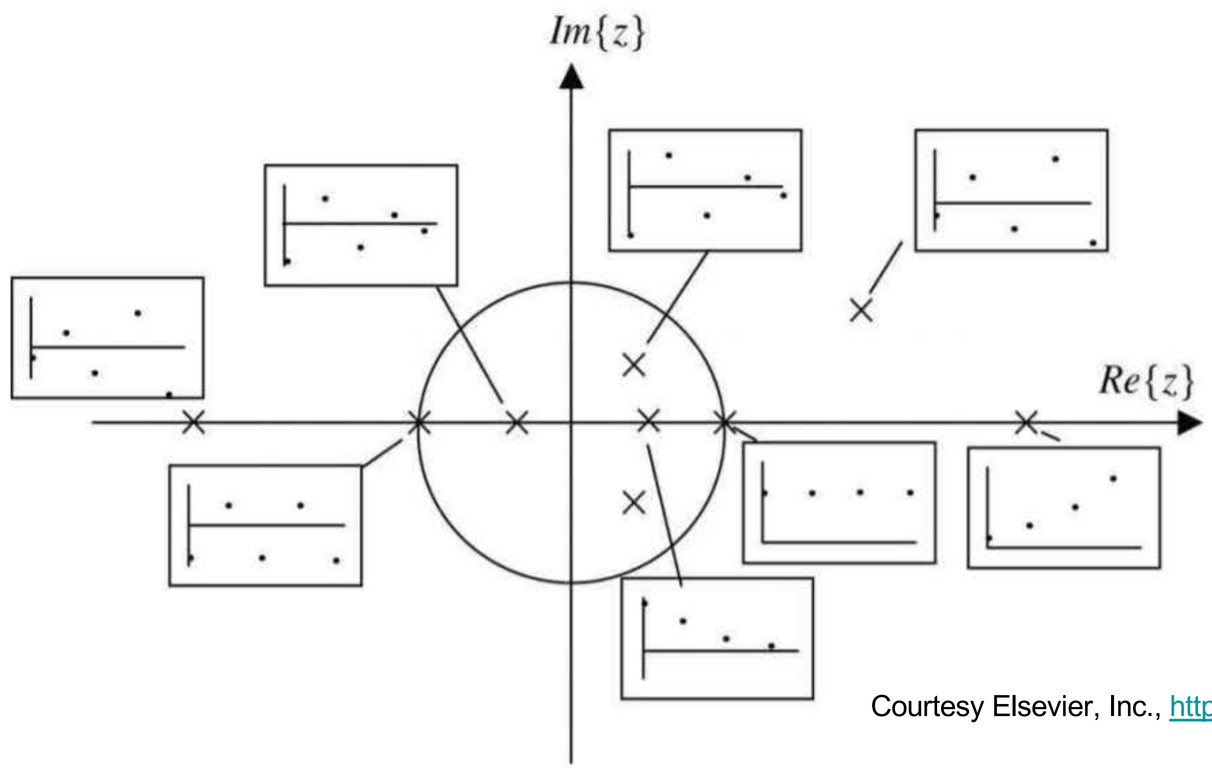
$\lambda_1, \lambda_2, \ldots, \lambda_n$ eigenvalues of $\mathbf{A}$

$\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_n}$ eigenvectors of $\mathbf{A}$

$c_1, c_2, \ldots, c_n$ constants depending on the initial condition

Digital traffic control

There is stability for eigenvalues that are inside the unit circle around the origin of the complex plane.



Source: M. Sami Fadali, and A. Visioli. Digital Control Engineering: Analysis and Design. 3rd ed. Academic Press, 2020.

References

T. Phan, A. M. Annaswamy, D. Yanakiev, and E. Tseng, “A model-based dynamic toll pricing strategy for controlling highway traffic,” in Proc. Amer. Control Conf. (ACC), 2016, pp. 6245–6252.

M. Sami Fadali, and A. Visioli. Digital Control Engineering: Analysis and Design. 3rd ed. Academic Press, 2020.