18.085/18.0851 Computational Science and Engineering I **Summer 2020**

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In the past few weeks, we have learned how to solve basic differential equations in $1D$. Now we shall move to multi-dimensions. To do that, we will need a few key tools in multi-dimensional calculus.

6.1 Multi-Dimensional Differentiation

In multi-dimensional calculus, we are concerned with functions that either take in multiple values or output multiple values, or both. Mathematically, we are concerned with functions $f : \mathbb{R}^m \to \mathbb{R}^n$. Examples include

- $f(x, y) = x^2 + y^2$. $f : \mathbb{R}^2 \to \mathbb{R}$
- $f(x, y, z) = [x, y^2, xy, \sqrt{xyz}]^T . f : \mathbb{R}^3 \to \mathbb{R}^4$

There are four basic operations that we would like to study; gradient, divergence, curl, and Laplacian.

6.1.1 Gradient

Denoted as ∇, the gradient of a function, $f : \mathbb{R}^n \to \mathbb{R}$, is defined as

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \dots \\ \frac{\partial f}{\partial x_n} \end{bmatrix} \tag{6.1}$$

As an example, suppose we have $f(x, y, z, a) = x^3 y + 2ya + z + \sqrt{a}$, the gradient of f is defined as

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \\ \frac{\partial f}{\partial a} \end{bmatrix} \tag{6.2}$$

$$= \begin{bmatrix} 3x^2 y \\ x^3 + 2a \\ 1 \\ \frac{1}{2\sqrt{a}} + 2y \end{bmatrix} \tag{6.3}$$

We can evaluate the gradient at $(x, y, z, a) = (1, 0, -1, 2)^T$, we will have that

$$\nabla f = \begin{bmatrix} 0 \\ 5 \\ 1 \\ \frac{1}{2\sqrt{2}} \end{bmatrix} \tag{6.4}$$

For a given surface, the normal unit vector, $\hat{n}$, is defined as the vector perpendicular to the surface. The normal derivative of a function f is defined as

$$\frac{\partial f}{\partial n} = \nabla f \cdot \hat{n} \tag{6.5}$$

In other words, we are looking at the weighted sum of derivatives that are normal to the surface.

6.1.2 Divergence

Denoted as $\nabla\cdot$ or just *div*, the divergence of a function $f : \mathbb{R}^n \to \mathbb{R}^n$ is defined as

$$\nabla \cdot f = div(f) \tag{6.6}$$
$$= \sum_{i=1}^{n} \frac{\partial f_n}{\partial x_n} \tag{6.7}$$

For example, suppose we have $f(x, y, z) = (xy, yz, xz^2)$. The divergence of f is then

$$\nabla \cdot f = div(f) \tag{6.8}$$
$$= \frac{df_x}{dx} + \frac{df_y}{dy} + \frac{df_z}{dz} \tag{6.9}$$
$$= y + z + 2xz \tag{6.10}$$

The divergence of f evaluated at $(x, y, z) = (1, 1, 1)$ would be

$$\nabla \cdot f = 1 + 1 + 2 \tag{6.11}$$
$$= 4 \tag{6.12}$$

6.1.3 Curl

Denoted as $\nabla\times$, or just *curl*, the curl of a function $f : \mathbb{R}^3 \to \mathbb{R}^3$, is defined as

$$\nabla \times f = curl(f) \tag{6.13}$$
$$= \det\left(\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ f_x & f_y & f_z \end{bmatrix}\right) \tag{6.14}$$
$$= \hat{i}(\partial_y f_z - \partial_z f_y) - \hat{j}(\partial_x f_z - \partial_z f_x) + \hat{k}(\partial_x f_y - \partial_y f_x) \tag{6.15}$$

As an example, suppose that $f(x, y, z) = (xy^2, xz^3, yxz)$, then the curl of f is

$$\nabla \times f = \hat{i}(xz - 3z^2x) - \hat{j}(yz - 0) + \hat{k}(z^3 - 2xy) \tag{6.16}$$

6.1.4 Laplacian

Denoted as ∇^2 or Δ, the Laplacian of a function is the divergence of the gradient of a function $f : \mathbb{R}^n \to \mathbb{R}$. For example, the Laplacian of $f(x, y, z, a) = x^3y + 2ya + z + \sqrt{a}$ is

$$\nabla^2 f = \Delta f \tag{6.17}$$
$$= 6xy + 0 + 0 - \frac{1}{4}a^{-3/2} \tag{6.18}$$
$$= 6xy - \frac{1}{4}a^{-3/2} \tag{6.19}$$

Two quick points

- When $n = 1$, the Laplacian is simply the second derivative of a function.
- $\Delta u = 0$ is known as the Laplace's equation. As one of the most fundamental partial differential equations, it governs many natural and engineering processes, such as electromagnetism, thermodynamics, fluid dynamics, etc.

6.2 Multi-Dimensional Integration

Just as multi-dimensional derivatives can get very creative, there are various ways we can perform integration in functions of several variables.

6.2.1 Double Integrals: Area and Volume

One of the most straightforward integration is over a certain area of volume. In $2D$, this would be an area integral. In $3D$, this would be a volume.

Let us integrate $f(x, y) = x^2 y$ over the area confined by $y = 0, x = 0$, and $x = y$. Then from the graph, we can deduct that y needs to go from 0 to x and x needs to go from 0 to 1. Hence we write

$$\int\int f(x,y)dxdy = \int_0^1 \int_0^x x^2 ydxdy \tag{6.20}$$
$$= \int_0^1 x^2 \int_0^x ydydx \tag{6.21}$$
$$= \int_0^1 x^2 \frac{x^2}{2} dx \tag{6.22}$$
$$= \frac{x^5}{10}\Big|_0^1 \tag{6.23}$$
$$= \frac{1}{10} \tag{6.24}$$

6.2.2 Line Integrals

The key to multi-dimensional integration is evaluating the multi-dimensional functions $f(x_1, x_2, ...x_n)$. As long as we can find a source of evaluating the function $f(x)$, we should be able to integrate the function over that source. That source, for instance, can be a line.

Suppose we are interested in integrate f along the curve parametrized by $r(t), a \leq t \leq b$. Then the line integral of f defined along the curve would be

$$\int_a^b f(t)|r'(t)|dt \tag{6.25}$$

As an example, suppose we have $f(x, y) = ye^x$ along C where C is the line segment between $(1, 2)$ and $(4, 7)$. Then we can write the line segment as a vector function $v = (1, 2) + t(3, 5)$, where $0 \leq t \leq 1$. In this way,

$x = 1 + 3t$ and $y = 2 + 5t$. Hence

$$\int_C f(x,y)dl \tag{6.26}$$

$$= \int_0^1 (2+5t)e^{1+3t}\sqrt{3^2+5^2}dt \tag{6.27}$$

$$= \frac{16}{9}\sqrt{34}e^4 - \frac{1}{9}\sqrt{34}e \tag{6.28}$$

6.2.3 Surface integrals

If the source is the surface element of an area, then we can have a surface integral. Suppose the surface is defined as $g(x, y)$. Then the surface integral over S would be

$$\int_S f(x,y)dS = \int_S f(x,y)\sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}dxdy \tag{6.29}$$

We will illustrate the technical calculations of surface integrals later on in the context of Divergence Theorem

6.3 Divergence Theorem and Stokes' Theorem

The two most important theorems in vector calculus are divergence theorem and Stokes' theorem.

6.3.1 Divergence Theorem

Suppose we have a function $F(x, y, z) : \mathbb{R}^3 \to \mathbb{R}^3$. Then over a volume V enclosed by a surface S, we can write down the relation

$$\int\int\int_V \nabla \cdot F dxdydz = \oint F \cdot \hat{n} dS \tag{6.30}$$

As an example, let's verify the divergence theorem of $F(x, y) = (5x, 2y)$ over the "volume" $R = \{(x, y) : x^2 + y^2 \leq 1\}$. On the one hand, we have

$$\int_R \nabla \cdot F dxdy = \int_R (5+2)dxdy \tag{6.31}$$

$$= 7\int_R dxdy \tag{6.32}$$

$$= 7\pi \tag{6.33}$$

On the other hand, along the curve that confines the "volume", the normal unit vector $\hat{n}$ is defined as

$(\cos(\theta), \sin(\theta))$, where $0 \le \theta \le 2\pi$. Then we can compute the flux

$$\begin{aligned}
\int_S F \cdot \hat{n} dS &= \int_0^{2\pi} (5\cos(\theta), 2\sin(\theta)) \cdot (\cos(\theta), \sin(\theta) d\theta && (6.34)\\
&= \int_0^{2\pi} (2\cos^2(\theta) + 5\sin^2(\theta)) d\theta && (6.35)\\
&= 2\int_0^{2\pi} \cos^2(\theta) d\theta + 5\int_0^{2\pi} \sin(\theta) d\theta && (6.36)\\
&= 2\pi + 5\pi && (6.37)\\
&= 7\pi && (6.38)
\end{aligned}$$

6.3.2 Stokes Theorem

If $F : \mathbb{R}^3 \to \mathbb{R}^3$, then over any surface S, not necessarily closed, we have

$$\oint_C F \cdot dl = \int \nabla \times F dA \tag{6.39}$$

You will get to verify Stokes' theorem in a homework problem

6.4 Fourier Series

The idea of Fourier series comes from a wish to mathematically represent a periodic function. For instance,

$$f(x) = \begin{cases} -1, & -\pi \le x < 0 \\ 1, & 0 \le x \le \pi \end{cases} \tag{6.40}$$

and now imagine that I extend the function of period 2π. It is easy to describe them in English, but how do we do so mathematically?

6.4.1 Functional Space and Basis Vectors

We spoke earlier about the idea of extending the notion of vector space in linear algebra to functions. We can speak of the collection of functions as a vector space, over which each individual function is a vector. Then immediately, we can carry over many concepts from linear algebra to calculus. Here is a list

- Vector Space $\mathbb{R}^n \iff$ functional space V over $[-\pi, \pi]$
- Vectors ("pointy arrows") in $\mathbb{R}^n \iff$ a function $f(x)$ in V
- Dot product between two vectors in $\mathbb{R}^n \iff$ inner product over $[-\pi, \pi]$ between two functions in V
- Basis vectors (eg. standard basis $e_1, e_2, ...e_n$) in $\mathbb{R}^n \iff$ basis functions
- Linear combination of basis vectors forming a vector in $\mathbb{R}^n \iff$ Fourier series

6.4.1.1 Inner Product

For any given two functions, $f(x)$ and $g(x)$, the inner product over the interval (a, b) is a scalar value defined as

$$\langle f, g\rangle = \int_a^b f(x)g(x)dx \tag{6.41}$$

Two comments

- If $f = g$, then $\langle f, f\rangle = \int_a^b f^2 dx = ||f||_{L^2(a,b)}$ is known as the L^2 norm of f over the interval $[a, b]$
- If $\langle f, g\rangle = 0$, then f and g are called orthogonal functions. This is consistent with the orthogonality definition of dot products in $\mathbb{R}^n$.

6.4.1.2 Basis Functions and Linear Combinations

For the function space V over the interval $[-\pi, \pi]$, a suitable collection of basis vectors would be: $\sin(nx)$ and $\cos(nx)$, for $n = 0, 1, 2, 3, ...$

In other words, the claim is that all functions can be represented as a linear combination of $\sin(nx)$ and $\cos(nx)$, ie. there exists a_n, b_n, for any given $f(x)$ over $[-\pi, \pi]$, such that

$$f(x) = \sum_{i=0}^{\infty} a_n \cos(nx) + \sum_{i=1}^{\infty} b_n \sin(nx) \tag{6.42}$$

This is known as the Fourier series representation of $f(x)$. Note that

- $\int_{-\pi}^{\pi} \sin(nx)\cos(kx)dx = 0$
- $\int_{-\pi}^{\pi} \sin(nx)\sin(kx)dx = \delta_{nk}\pi$
- $\int_{-\pi}^{\pi} \cos(nx)\cos(kx)dx = \delta_{nk}\pi$

where

$$\delta_{nk} = \begin{cases} 0, & n \neq k \\ 1, & n = k \end{cases} \tag{6.43}$$

This will be left as a homework. In any case, this shows that $\sin(nx)$ and $\cos(kx)$ are orthogonal basis vectors.

6.4.1.3 Calculate the Fourier coefficients

The question remains how we can find the a_n and b_n. To do that, we continue our analogy with linear algebra. Suppose we have a vector $v = (-3, 4)^T$. We want to decompose into a linear combination of orthogonal basis vectors v_1, v_2, where

$$v_1 = (1, 1)^T \tag{6.44}$$

$$v_2 = (1, -1)^T \tag{6.45}$$

Then there exists $a_1, a_2 \in \mathbb{R}$, such that

$$v = a_1 v_1 + a_2 v_2 \tag{6.46}$$

The way we find a_1 and a_2 are the following

- Take the dot product of v_1 on both sides. Note that the v_2 term would then vanish due to orthogonality
- Solve for a_1
- Repeat the above process for a_2

Hence,

$$\begin{aligned} v \cdot v_1 &= a_1 v_1 \cdot v_1 + a_2 v_2 \cdot v_1 && (6.47) \\ &= a_1 v_1 \cdot v_1 && (6.48) \\ a_1 &= \frac{v \cdot v_1}{v_1 \cdot v_1} && (6.49) \end{aligned}$$

Similarly,

$$a_2 = \frac{v \cdot v_2}{v_2 \cdot v_2} \tag{6.50}$$

we calculate that

- $v \cdot v_1 = -3 + 4 = 1$
- $v \cdot v_2 = -7$
- $v_1 \cdot v_1 = 2$
- $v_2 \cdot v_2 = 2$

Therefore, $a_1 = 1/2$ and $a_2 = -7/2$

The same procedure should apply to Fourier series of f:

- Take the inner product on both sides with respect to $\sin(kx)$. All terms that are not of $\sin(kx)$ should disappear
- Solve for b_k
- Repeat the above process for a_k

Therefore, if

$$f(x) = \sum_{i=0}^{\infty} a_n \cos(nx) + \sum_{i=1}^{\infty} b_n \sin(nx) \tag{6.51}$$

Then

$$\langle f(x), \cos(kx)\rangle = \left\langle \sum_{i=0}^{\infty} a_n \cos(nx) + \sum_{i=1}^{\infty} b_n \sin(nx), \cos(kx) \right\rangle \tag{6.52}$$

$$\int_{-\pi}^{\pi} f(x)\cos(kx)dx = \int_{-\pi}^{\pi} \left(\sum_{i=0}^{\infty} a_n \cos(nx)\cos(kx) + \sum_{i=1}^{\infty} b_n \sin(nx) \right) \cos(kx)dx \tag{6.53}$$

$$= \sum_{i=0}^{\infty} a_n \int_{-\pi}^{\pi} \cos(nx)\cos(kx)dx + \sum_{i=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin(nx)\cos(kx)dx \tag{6.54}$$

$$= \sum_{i=0}^{\infty} a_n \int_{-\pi}^{\pi} \cos(nx)\cos(kx)dx \tag{6.55}$$

$$= a_k \int_{-\pi}^{\pi} \cos^2(kx)dx \tag{6.56}$$

$$a_k = \frac{1}{\int_{-\pi}^{\pi} \cos^2(kx)dx} \int_{-\pi}^{\pi} f(x)\cos(kx)dx \tag{6.57}$$

Note that if $k > 0$, $\int_{-\pi}^{\pi} \cos^2(kx)$. If $k = 0$, $\int_{-\pi}^{\pi} \cos^2(kx) = 2\pi$. Hence

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx \tag{6.58}$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(kx)dx \tag{6.59}$$

Following the same procedure, we can get that

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\sin(kx)dx \tag{6.60}$$

Note that b_0 does not exist because $\sin(0) = 0$

6.4.2 A Concrete Example

Now let us do a concrete example. Suppose we have that $f(x)$ is defined as

$$f(x) = \begin{cases} -1, & -\pi \le x < 0 \\ 1, & 0 \le x \le \pi \end{cases} \tag{6.61}$$

We will lay out a few steps to do so

6.4.2.1 Step 1: Identify the periods and write down the formula

It may seem trivial, but if the function is not given algebraically, it may not be clear what the period is immediately. So we have to identify what the periods are and write them down. In this case, it is clear that the period is 2π over $[-\pi, \pi]$. Hence we know the previous calculations that

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \tag{6.62}$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx \tag{6.63}$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx \tag{6.64}$$

6.4.2.2 Step 2: Observe even/oddness

Since $\sin(nx)$ and $\cos(nx)$ are odd and even functions, respectively, we can make an argument about the event/oddness of $f(x)$ to eliminate the corresponding terms in the Fourier series.

- If $f(x)$ is even, then $b_k = 0$ as there can be no odd terms
- IF $f(x)$ is odd, then $a_k = 0$ as there can be no even terms.

In this case, because $f(x)$ is odd, $a_k = 0$, whence we only have the coefficients b_k.

6.4.2.3 Step 3: Compute the integrals

We do the computation after we have eliminated the obvious.

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx \tag{6.65}$$

$$= -\frac{1}{\pi}\int_{-\pi}^{0} \sin(kx)dx + \frac{1}{\pi}\int_{0}^{\pi} \sin(kx)dx \tag{6.66}$$

$$= -\frac{-\cos(kx)}{k\pi}|_{-\pi}^{0} + \frac{-\cos(kx)}{k\pi}|_{0}^{\pi} \tag{6.67}$$

$$= \frac{\cos(kx)}{k\pi}|_{-\pi}^{0} - \frac{\cos(kx)}{k\pi}|_{0}^{\pi} \tag{6.68}$$

$$= \frac{1-(-1)^k}{k\pi} - \frac{(-1)^k - 1}{k\pi}|_{0}^{\pi} \tag{6.69}$$

$$= \frac{2(1-(-1)^k)}{k\pi} \tag{6.70}$$

Observe that for k even, $a_k = 0$. Hence,

$$b_k = \frac{4}{k\pi} \tag{6.71}$$

Hence, the Fourier series of of $f(x)$ is

$$f(x) = \sum_{n=1,3,5,\ldots}^{\infty} \frac{4}{k\pi}\sin(kx) \tag{6.72}$$

If we observe the convergence plot, we notice that there is a lot of oscillations. This is known as the Gibbs phenomenon.

6.4.3 Complex Exponential

We know that

$$f(x) = \sum_{i=0}^{\infty} a_n \cos(nx) + \sum_{i=1}^{\infty} b_n \sin(nx) \tag{6.73}$$

But since sin and cos can be expressed in terms of complex exponential, we can re-express $f(x)$, a function over the interval $[-\pi, \pi]$ as a linear combination of e^{inx}. In other words,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \tag{6.74}$$

6.4.4 Inner product

Because the basis function is complex, we will need to redefine the notion of inner product

$$\langle f, g \rangle = \int_{-\pi}^{\pi} \bar{f} g dx \tag{6.75}$$

Here $\bar{f}$ is the complex conjugate of f.
We prove that $\{e^{inx}\}_{n=-\infty}^{\infty}$ is a set of orthogonal basis vectors. If $k \neq n$

$$\begin{aligned}
\left\langle e^{inx}, e^{ikx} \right\rangle &= \int_{-\pi}^{\pi} e^{-inx} e^{ikx} dx && (6.76)\\
&= \frac{e^{i(k-n)x}}{i(k-n)} |_{-\pi}^{\pi} && (6.77)\\
&= \frac{\cos(k-n)\pi + i\sin(k-n)\pi - \cos(k-n)\pi + i\sin(k-n)}{i(k-n)} && (6.78)\\
&= 0 && (6.79)
\end{aligned}$$

If $n = k$, then

$$\begin{aligned}
\left\langle e^{ikx}, e^{ikx} \right\rangle &= \int_{-\pi}^{\pi} e^{-ikx} e^{ikx} dx && (6.80)\\
&= \int_{-\pi}^{\pi} dx && (6.81)\\
&= 2\pi && (6.82)
\end{aligned}$$

Henceforth, we can use the same "Fourier trick" as before, exploiting the orthogonality of the basis functions to compute c_n.

$$\begin{aligned}
\left\langle e^{ikx}, f(x) \right\rangle &= \left\langle \sum_{n=-\infty}^{\infty} e^{ikx}, c_n e^{inx} \right\rangle && (6.83)\\
&= \sum_{n=-\infty}^{\infty} c_n \left\langle e^{ikx}, e^{inx} \right\rangle && (6.84)\\
&= c_k(2\pi) && (6.85)\\
c_k &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx && (6.86)
\end{aligned}$$

Let's redo the example of the step function $f(x)$

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx} \tag{6.87}$$
$$= \frac{1}{2\pi}\int_{-\pi}^{0} e^{-ikx}dx + \frac{1}{2\pi}\int_{0}^{\pi} e^{-ikx}dx \tag{6.88}$$
$$= \frac{1}{2\pi}\frac{e^{-ikx}}{ik}|_{-\pi}^{0} - \frac{1}{2\pi}\frac{e^{-ikx}}{ik}|_{0}^{\pi} \tag{6.89}$$
$$= \frac{1}{2\pi}\frac{1-e^{ik\pi}}{ik} - \frac{1}{2\pi}\frac{e^{-ik\pi}-1}{ik} \tag{6.90}$$
$$= \frac{1}{2\pi}\frac{2-e^{ik\pi}-e^{-ik\pi}}{ik} \tag{6.91}$$
$$= \frac{1}{2\pi}\frac{2-2\cos(kx)}{ik} \tag{6.92}$$
$$= \frac{1}{\pi}\frac{1-\cos(kx)}{ik} \tag{6.93}$$
$$= \frac{1}{\pi}\frac{1-(-1)^k}{ik} \tag{6.94}$$

Hence $c_k = \frac{2}{ik\pi}$ if k is even and 0 if k is odd. Hence

$$f(x) = \sum_{n=-\infty,odd}^{\infty} \frac{2}{ik\pi}e^{ikx} \tag{6.95}$$

We group the positive and negative terms of corresponding degree together and obtain that

$$f(x) = \sum_{k=1,3,5,\dots}^{\infty} \frac{2}{ik\pi}\left(e^{ikx} - e^{-ikx}\right) \tag{6.96}$$
$$= \sum_{k=1,3,5,\dots}^{\infty} \frac{2}{ik\pi}\left(e^{ikx} - e^{-ikx}\right) \tag{6.97}$$
$$= \sum_{k=1,3,5,\dots}^{\infty} \frac{2}{k\pi}\left(2\sin(kx)\right) \tag{6.98}$$
$$= \sum_{k=1,3,5,\dots}^{\infty} \frac{4}{k\pi}\sin(kx) \tag{6.99}$$

So we recover the result using sines and cosines.

6.4.5 Other Technicalities

6.4.5.1 Even Functions

If the function that I am expanding over is even, ie. $f(x) = f(-x)$, then immediately I know that all $b_n = 0$. This is because the $sin(nx)$ is an odd function that would have made f odd if they ever show up in the series expansion. Furthermore, because f is even,

$$\int_{-\pi}^{\pi} f(x)\cos(nx) = 2\int_{0}^{\pi} f(x)\cos(nx) \tag{6.100}$$

6.4.5.2 Odd Functions

Similarly, if the function $f(x)$ is odd, ie $f(x) = -f(x)$, then immediately I know that all $a_n = 0$ since any existence would a cosine term would make the function even. Furthermore, because $f(x)\sin(nx)$ is even,

$$\int_{-\pi}^{\pi} f(x)\sin(nx) = 2\int_{0}^{\pi} f(x)\sin(nx) \tag{6.101}$$

6.4.5.3 Fourier Series over $[-L, L]$

To Fourier expand a function over $[-L, L]$, we must modify the basis function to be $\sin\left(\frac{n\pi x}{L}\right)$ and $\cos\left(\frac{n\pi x}{L}\right)$, so that the Fourier seires will become

$$f(x) = \sum_{n=0}^{\infty} a_n \cos\frac{n\pi}{L}x + \sum_{n=0}^{\infty} b_n \sin\frac{n\pi}{L}x \tag{6.102}$$

You will explore in the homework how to modify the expression for a_n and b_n.