5.6 Nonlinear Flow and Conservation Laws

Nature is nonlinear. The coefficients in the equation depend on the solution \( u \). In place of \( u_t = c u_x \) we will study \( u_t + uu_x = 0 \) and more generally \( u_t + f(u)_x = 0 \). These are “conservation laws” and the conserved quantity is the integral of \( u \).

The first part of this book emphasized the balance equation: forces balance and currents balance. For steady flow this was Kirchhoff’s Current Law: flow in equals flow out. The net flow was zero. Now the flow is unsteady—the “mass inside” is changing. So a new \( \partial / \partial t \) term will enter the conservation law.

There is “flux” through the boundaries. In words, the rate of change of mass inside a region equals that incoming flux. For an interval \([a, b]\), the incoming flux is the difference in fluxes at the endpoints \( a \) and \( b \):

\[
\begin{align*}
\text{Integral form} & \quad \frac{d}{dt} \int_a^b u(x, t) \, dx = f(u(a, t)) - f(u(b, t)) .
\end{align*}
\]

In applications, \( u \) can be a density (of cars along a highway). The integral of \( u \) gives the mass (number of cars) between \( a \) and \( b \). This number changes with time, as cars flow in at point \( a \) and out at point \( b \). The flux is density \( u \) times velocity \( v \).

The integral form is fundamental. We can get a differential form by allowing \( b \) to approach \( a \). Suppose \( b - a = \Delta x \). If \( u(x, t) \) is a smooth function, its integral over a distance \( \Delta x \) will have leading term \( \Delta x u(a, t) \). So if we divide equation (1) by \( \Delta x \), the limit as \( \Delta x \) approaches zero is \( \partial u / \partial t = -\partial f(u) / \partial x \):

\[
\begin{align*}
\text{Differential form} & \quad \frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = \frac{\partial u}{\partial t} + f'(u) \frac{\partial u}{\partial x} = 0 .
\end{align*}
\]

When \( f(u) = \text{density} u \times \text{velocity} v(u) \), we can solve this single conservation law. For traffic flow, the velocity \( v(u) \) can be measured (it will decrease as density increases). In gas dynamics there are also conservation laws for momentum and energy. The velocity \( v \) becomes another unknown, along with the pressure \( p \). The Euler equations for gas dynamics in one space dimension include two additional equations:

\[
\begin{align*}
\text{Conservation of momentum} & \quad \frac{\partial}{\partial t} (uv) + \frac{\partial}{\partial x} (uv^2 + p) = 0 .
\end{align*}
\]

\[
\begin{align*}
\text{Conservation of energy} & \quad \frac{\partial}{\partial t} (E) + \frac{\partial}{\partial x} (Ev + Ep) = 0 .
\end{align*}
\]

Systems of conservation laws are more complicated, but our scalar equation (2) already has the possibility of shocks. A shock is a discontinuity in the solution \( u(x, t) \), where the differential form breaks down and we need the integral form (1).
The other outstanding example, together with traffic flow, is Burger’s equation, for \( u = \) velocity. The flux \( f(u) = \frac{1}{2} u^2 \). The “inviscid” form has no \( u_{xx} \):

\[
\text{Burger’s equation} \quad \frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left( \frac{u^2}{2} \right) = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0.
\]

When both the density and velocity are unknowns, these examples combine into conservation of mass and conservation of momentum. Typically we change density to \( \rho \). For small disturbances of a uniform density \( \rho_0 \), we could linearize the conservation laws and reach the wave equation (Problem 5.6). But the Euler and Navier-Stokes equations are truly nonlinear, and we begin the task of solving them.

We will approach conservation laws (and these examples) in three ways:

1. By following characteristics until trouble arrives: they separate or collide
2. By a special formula ( )
3. By finite difference and finite volume methods, which are the practical choice.

**Characteristics**

The one-way wave equation \( u_t = c u_x \) is solved by \( u(x, t) = u(x + ct, 0) \). Every initial value \( u_0 \) is carried along a characteristic line \( x + ct = x_0 \). Those lines are parallel when the velocity \( c \) is a constant.

The conservation law \( u_t = +u u_x = 0 \) will be solved by \( u(x, t) = u(x - ut, 0) \). Every initial value \( u_0 = u(x_0, 0) \) is carried along a characteristic line \( x - u_0 t = x_0 \). Those lines are not parallel because their slopes depend on the initial value \( u_0 \).

Notice that the formula \( u(x, t) = u(x - ut, 0) \) involves \( u \) on both sides. It gives the solution “implicitly.” If the initial function is \( u(x, 0) = 1 - x \), for example, the formula must be solved for \( u \):

\[
u = 1 - (x - ut) \quad \text{gives} \quad (1 - t)u = 1 - x \quad \text{and} \quad u = \frac{1 - x}{1 - t}.
\]

This does solve Burger’s equation, since the time derivative \( u_t = (1 - x)/(1 - t)^2 \) is equal to \(-u u_x \). The characteristic lines (with different slopes) can meet. This is an extreme example, where all characteristics meet at the same point:

\[
x - u_0 t = x_0 \quad \text{or} \quad x - (1 - x_0) t = x_0 \quad \text{which goes through} \ x = 1, t = 1 \quad (6)
\]

You see how the solution \( u = (1 - x)/(1 - t) \) becomes 0/0 at that point \( x = 1, t = 1 \). Beyond their meeting point, the characteristics cannot completely decide \( u(x, t) \).

A more fundamental example is the Riemann problem, which starts from two constant values \( u = A \) and \( u = B \). Everything depends on whether \( A > B \) or \( A < B \). On the left side of Figure 5.13, with \( A > B \), the characteristics meet. On the right
side, with $A < B$, the characteristics separate. Both cases present a new (nonlinear) problem, when we don’t have a single characteristic that is safely carrying the correct initial value to the point. This Riemann problem has two characteristics through the point, or none:

**Shock**  Characteristics collide  (light goes red: speed drops from 60 to 0)

**Fan**  Characteristics separate  (light goes green: speed up from 0 to 60)

The problem is how to connect $u = 60$ to $u = 0$, when the characteristics don’t give the answer. A shock will be sharp breaking (drivers only see the car ahead in this model). A fan will be gradual acceleration.

TO DO...

Figure 5.13: A shock when characteristics collide, a fan when they separate.

For the conservation law $u_t + f(u)_x = 0$, the characteristics are $x - f'(u_0)t = x_0$. That line has the right slope to carry the constant value $u = u_0$:

$$
\frac{d}{dt} u(x_0 + St, t) = S \frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0 \quad \text{when} \quad S = f'(u). \quad (7)
$$

The solution until trouble arrives is $u(x, t) = u(x - f'(u)t, 0)$.

**Shocks**

After trouble arrives, it will be the integral form that guides the choice of the correct solution $u$. If there is a jump in $u$ (a shock), that integral from tells where the jump must occur. Suppose $u$ has different values $u_L$ and $u_R$ at points $x_L$ and $x_R$ on the left and right sides of the shock:

**Integral form**

$$
\frac{d}{dt} \int_{x_L}^{x_R} u \, dx + f(u_R) - f(u_L) = 0. \quad (8)
$$

If the position of the shock is $x = X(t)$, we take $x_L$ and $x_R$ very close to $X$. The values of $u(x, t)$ inside the integral are close to the constants $u_L$ and $u_R$:

$$
\frac{d}{dt} \left[ (x - x_L) u_L + (x_R - X) u_R \right] + f(u_R) - f(u_L) \approx 0.
$$

This gives the speed $s = dX/dt$ of the shock curve:

**Jump condition**

$$
su_L - su_R + f(u_R) - f(u_L) = 0
$$

**Shock speed**

$$
\frac{f(u_R) - f(u_L)}{u_R - u_L} = \left[ \frac{f}{u} \right]. \quad (9)
$$
For the Riemann problem, the left and right values \( u_L \) and \( u_R \) will be constants \( A \) and \( B \). The shock speed \( s \) is the ratio between the jump \([f]\) = \( f(B) - f(A) \) and the jump \([u]\) = \( B - A \). Since this ration gives a constant slope, the shock line is straight. For other problems, the characteristics are carrying different values of \( u \) into the shock. So the shock speed \( s \) is not constant and the shock lines is curved.

The shock gives the solution when characteristics collide. With \( f(u) = \frac{1}{2} u^2 \) in Burger’s equation, the shock speed is halfway between \( u_L \) and \( u_R \):

\[
\text{Burger’s equation} \quad \text{Shock speed} \ s = \frac{1}{2} \frac{u_R^2 - u_L^2}{u_R - u_L} = \frac{1}{2} (u_R + u_L). \tag{10}
\]

The Riemann problem has \( u_L = A \) and \( u_R = B \), and \( s \) is their average. Figure 5.14 shows how the integral form of Burger’s equation is solved by the right placement of the shock.

\section*{Fans}

You might expect a similar picture (just flipped) when \( A < B \). Wrong. The integral form is still satisfied, but it is also satisfied by a fan. The choice between shock and fan is made by the “entropy condition” that as \( t \) increases, characteristics must go into the shock. The wave speed is faster than the shock speed on the left, and slower on the right:

\[
\text{Entropy condition} \quad f'(u) > s > f'(u_R). \tag{11}
\]

Since Burger’s equation has \( f'(u) = u \), it only has shocks when \( u_L \) is larger than \( u_R \). In the Riemann problem that means \( A > B \). In the opposite case, the smaller value \( u_L = A \) has to be connected to \( u_R = B \) by the fan in Figure 5.14:

\[
\text{Fan (or rarefaction)} \quad u = \frac{x}{t} \quad \text{for} \quad At < x < Bt. \tag{12}
\]

use fig 6.28 p. 592 of IAM (reverse left and right figs)

Figure 5.14: Characteristics collide in a shock and separate in a fan.

Notice especially that in the traffic flow problem, the velocity \( v(u) \) decreases as the density \( u \) increases. A good model is linear between \( v = v_{max} \) at zero density and \( v = 0 \) at maximum density. Then the flux \( f(u) = uv(u) \) is a downward parabola (concave instead of Burger’s convex \( u^2/2 \)):

\[
\text{Traffic speed and flux} \quad v(u) = v_{max} \left(1 - \frac{u}{u_{max}}\right) \quad \text{and} \quad f(u) = v_{max} \left(u - \frac{u^2}{u_{max}}\right). \tag{13}
\]

Typical values for a single lane of traffic show a maximum flux of \( f = 1600 \) vehicles per hour, when the density is \( u = 80 \) vehicles per mile. This maximum flow rate
is attained when the velocity \( f/u \) is \( v = 20 \) miles per hour! Small comfort at that speed, to know that other cars are getting somewhere too.

Problems ____ and ____ compute the solution when a light goes red (shock travels backward) and when a light goes green (fan moves forward). Please look at the figures, to see how the vehicle trajectories are entirely different form the characteristics.

A driver keeps adjusting the density to stay safely behind the car in front. (Hitting the car would give \( u < 0 \).) We all recognize the frustration of braking and accelerating from a series of shocks and fans. This traffic crawl happens when the green light is too short for the shock to make it through.

### A Solution Formula for Burger’s Equation

Let me comment on three nonlinear equations. They are useful models, quite special because each one has an exact solution formula:

| Conservation law | \( u_t + u u_x = 0 \) |
| Burger’s with viscosity | \( u_t + u u_x = \nu u_{xx} \) |
| Korteweg-de Vries | \( u_t + u u_x = -a u_{xxx} \) |

The conservation law can develop shocks. This won’t happen in the second equation because the \( u_{xx} \) viscosity term prevents it. That term can stay small when the solution is smooth, but it dominates when a wave is about to break. The profile is steep but it stays smooth.

As starting function for the conservation law, I will pick a point source: \( u(x, 0) = \delta(x) \). We can guess a solution with a shock, and check the jump condition and entropy condition. Then we find an exact formula when \( \nu u_{xx} \) is included, by a neat change of variables that produces \( h_t = \nu h_{xx} \). When we let \( \nu \rightarrow 0 \), the limiting formula solves the conservation law—and we can check that the following solution is correct.

**Solution with \( u(x, 0) = \delta(x) \)** When \( u(x,0) \) jumps upward, we expect a fan. When it drops we expect a shock. The delta function is an extreme case (very big jumps up and down, very close together!). So we look for a shock curve \( x = X(t) \) immediately in front of a fan!

\[
\text{Expected solution} \quad u(x,t) = \frac{x}{t} \text{ for } 0 \leq x \leq X(t); \quad \text{otherwise } u = 0. \tag{14}
\]

The total mass at the start is \( \int \delta(x) \, dx = 1 \). This never changes, and already that locates the shock position \( X(t) \):

\[
\text{Mass at time } t = \int_0^X \frac{x}{t} \, dt = \frac{X^2}{2t} = 1 \quad \text{so } X(t) = \sqrt{2t}. \tag{15}
\]
Does the drop in \( u \), from \( X/t = \sqrt{2t}/t \) to zero, satisfy the jump condition?

\[
\text{Shock speed } s = \frac{dX}{dt} = \frac{\sqrt{2}}{2\sqrt{t}} \text{ equals } \frac{\text{Jump } [u^2/2]}{\text{Jump } [u]} = \frac{X^2/2t^2}{X/t} = \frac{\sqrt{2t}}{2t}.
\]

The entropy condition \( u_L > s > u_R = 0 \) is also satisfied, and the solution \( (\ldots) \) looks good. It is good, but because of the delta function we check it another way.

Begin with \( u_t + u u_x = \nu u_{xx} \), and solve that equation exactly. If \( u(x) \) is \( \partial U/\partial x \), then integrating our equation gives \( U_t + \frac{1}{2} U_x^2 = \nu U_{xx} \). The initial value \( U_0(x) \) is now a step function. Then the great change of variables \( U = -2\nu \log h \) produces the heat equation \( h_t = \nu h_{xx} \) (Problem__). The initial value becomes \( h(x,0) = e^{-U_0(x)/2\nu} \).

Section 5.4 found the solution to the heat equation \( u_t = u_{xx} \) from any starting function \( h(x,0) \) and we just change \( t \) to \( \nu t \):

\[
U(x,t) = -2\nu \log h(x,t) = -2\nu \log \left[ \frac{1}{\sqrt{4\pi \nu t}} \int_{-\infty}^{\infty} e^{-U_0(y)/2\nu} e^{-(x-y)^2/4\nu t} dy \right]. \tag{16}
\]

It doesn’t look easy to let \( \nu \to 0 \), but it can be done. That exponential has the form \( e^{-B(x,y)/2\nu} \). This is largest when \( B \) is smallest. An asymptotic method called “steepest descent” shows that as \( \nu \to 0 \), the bracketed quantity in (16) approaches \( c e^{-B_{\min}/2\nu} \). Taking its logarithm and multiplying by \( -2\nu \), (16) becomes \( U = B_{\min} \) in the limit:

\[
\lim_{\nu \to 0} U(x,t) = B_{\min} = \min_y \left[ U_0(y) + \frac{1}{2\nu} (x-y)^2 \right]. \tag{17}
\]

This is the solution formula for \( U_t + \frac{1}{2} U_x^2 = 0 \). Its derivative \( u = U_x \) solves the conservation law \( u_t + u u_x = 0 \). By including the viscosity \( \nu u_{xx} \) with \( \nu \to 0 \), we are finding the \( u(x,t) \) that satisfies the jump condition and the entropy condition.

**Example** Starting from \( u(x,0) = \delta(x) \), its integral \( U_0 \) is a step function. The minimum of \( B \) is either at \( y = x \) or at \( y = 0 \). Check each case:

\[
U(t,x) = B_{\min} = \min_y \left[ 0 \begin{array}{c} \text{for } x \leq 0 \\ 1 \text{for } x > 0 \end{array} \frac{(x-y)^2}{2t} \right] = \begin{cases} 0 & \text{for } x \leq 0 \\ x^2/2t & \text{for } 0 \leq x \leq \sqrt{2t} \\ 1 & \text{for } x \geq \sqrt{2t} \end{cases}
\]

The result \( u = dU/dx \) is \( 0 \) or \( x/t \) or \( 0 \). This agrees with our guess in equation \( (\ldots) \)—a fan rising from 0 and a shock back to 0 at \( x = \sqrt{2t} \).