1. What is a dynamical system?
   1.1. Any system that changes over time
      a) Continual: Orbits
      b) Discrete: Compound Interest

2. Iteration: A process that is repeated over and over again.
   a) The output is sent back into the function as the new input
   b) Notation: The n-th iteration is written as $F^n(x)$

2.2. Example 1: For compounding interest at a rate of 10% per year:
   $A_0 = 100$
   $A_1 = 110$
   $A_2 = 121$
   $A_3 = ....$
   a) Can be modeled by an iteration $I(x) = 1.1x$
      $A_0 = 100$
      $A_1 = I(100) = 1.1*100 = 110$
      $A_2 = I(121) = I(I(100)) = 121$
      $A_3 = I^3(A)$
   b) Can be evaluated in general using $A_n = (1.1)^n A$

2.3. Example 2: Finding Square Roots:
   a) Need an algorithm. To find $\sqrt{n}$:
      1) Make a guess $x$
      2) Average $x$ and $n/x$
      3) Use the result as your new guess
      4) Repeat until guess is good enough
   b) Proof of Method:
      Given $x, n > 0$, we have two cases:
      Case 1: $\sqrt{n} < x$
      Case 2: $\sqrt{n} > x$
      $n < x \sqrt{n}$
      $n > x \sqrt{n}$
      $n/x < \sqrt{n}$
      $n/x > \sqrt{n}$

      In either case, $\sqrt{n}$ is between $n$ and $n/G$, so by averaging, we narrow the range
      in which $\sqrt{n}$ can lie.
c) For n=10 and an initial guess x=1

<table>
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<th>Iteration</th>
<th>Guess</th>
<th>Guess^2</th>
<th>Average</th>
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<td>1</td>
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2.4. Changing from continuous to discrete:
   a) Continually-compounded Interest:
      \[ A(t) = A_0e^{kt} \] [t= time in years] \( \implies \ A_n = A_0e^{kn} \) [n= # of years]
      - Changed from continuous with an iteration \( I(x) = (e^k)x \)
   b) Planetary Orbits
      - Draw an imaginary plane in space and look at the points where the orbit intersects the plane
   c) Gains some simplicity at the expense of some information
      - Discrete, instead of continuous
      - No information about behavior in between iterations

3. Orbits:
   3.1. Informally - The outputs of an iteration listed in the order that they are achieved
   3.2. Formally – Given \( x_0 \in \mathbb{R} \), the orbit of \( x_0 \) under \( F \) is the sequence of points \( x_0, x_1, x_2 \) such that \( x_n = F^n(x_0) \)

   3.3. Useful things we can say about orbits:
      a) Limit as \( n \to \infty \)
         - \( S(x) \to \sqrt{n} \)
      b) Are there any patterns?

4. Types of Orbits:
   4.1. Fixed Points: **Definition**: \( F(x_0) = x_0 \), for some \( x_0 \)
      - If \( a=\sqrt{n} \), \( S(a)=a \)
   4.2. Periodic Orbits / Cycles: **Definition**: \( F^k(x_0) = x_0 \), for some \( k, x_0 \)
      - If \( F(x) = 5 - x \)
        \( F(5) = 0 \) \( F(0) = 5 \)
      - Called a 2-cycle
   b) Finding a k-cycle
      - Solve the equation \( F^k(x) = x \)
      - If \( F \) is a quadratic function, this has degree \( 2^k \). In general, that is impossible to solve exactly.
   c) Note that if \( F \) has a k-cycle, then it has cycles of length nk, for all integers n
• $F^{3k}(x_0) = F^k(F^{2k}(x_0)) = F^{2k}(x_0) = \ldots = x_0$
• Prime Period: $n=1$

4.3. Eventually Fixed: **Definition:** $F^k(x_0) = x^*$, for all $k$ sufficiently large
   a) $F(x) = x^2 - 1$, $x_0 = (\sqrt{5} + 1)/2$
      $(\sqrt{5} + 1)/2, 0, 0, 0, \ldots$  

4.4. Eventually Periodic: **Definition:** $F^m(x_0) = F^n(x_0)$, for some $m,n$ greater than 1
   a) $F(x) = x^2 - 1$, $x_0 = \sqrt{\sqrt{2} + 1}$
      $\sqrt{\sqrt{2} + 1}, \sqrt{2}, 1, 0, -1, 0, -1, 0, \ldots$

5. Computers
   5.1. Uses:
      a) Visualizing orbits
      b) Approximating values:
         • solving $F(x)=x$
         • finding square roots
   5.2. Shortfalls:
      a) Rounding Errors!
         • Table on page 23
         • If $x$ is close enough to zero, the computer will round to zero, and the orbit will become fixed, instead of remaining chaotic like it should.