1. Consider the following two real-valued functions on $[0,1]$.

\[ f(x) = \begin{cases} 1, & x = \frac{1}{n} \text{ for some } n \in \mathbb{N} \\ 0, & \text{otherwise} \end{cases}, \quad g(x) = \begin{cases} n, & x = \frac{1}{n} \text{ for some } n \in \mathbb{N} \\ 0, & \text{otherwise} \end{cases}. \]

Show (from the definition) that $f \in \mathcal{R}$ (i.e. $f$ is Riemann-integrable), with $\int_0^1 f(x) \, dx = 0$, but $g \notin \mathcal{R}$.

2. Problem 8, page 138 in Rudin.

3. A subset $N \subseteq \mathbb{R}$ is said to have measure 0 if, for each $\epsilon > 0$, there is a sequence (finite or countable) of balls $(B_n)$ with radii $r_n$ so that $N \subseteq \bigcup_n B_n$ and $\sum_n r_n < \epsilon$.

(a) Let $N$ be a set of measure 0 in $\mathbb{R}$. Prove that the complement $N^c$ is dense in $\mathbb{R}$.

(b) Show (from the definition) that the only open set that has measure 0 is $\emptyset$.

(c) Can measure 0 sets be closed? Non-compact? Dense?

4. Let $\mathcal{R}^1[a,b]$ denote the set of all functions $f : [a,b] \to \mathbb{R}$ with the property that $|f|$ is Riemann integrable. Define a function $d_1 : \mathcal{R}^1[a,b] \times \mathcal{R}^1[a,b] \to \mathbb{R}_+$ by

\[ d_1(f, g) = \int_a^b |f - g|. \]

(a) Show that if $f$ is Riemann integrable on $[a,b]$, then $f \in \mathcal{R}^1[a,b]$.

(b) Is the converse of (a) true? [Hint: think about the function $g$ that equals 1 on $\mathbb{Q}$ and 0 on $\mathbb{Q}^c$; find non-zero constants $a, b$ so that $|ag + b|$ is constant.]

(c) Prove that $d_1$ satisfies all the axioms of a metric on $\mathcal{R}^1[a,b]$ except that $d_1(f, g) = 0$ for some $f \neq g$.

(d) Assume that $f, g$ are continuous on $[a,b]$. Prove that $d_1(f, g) = 0$ iff $f = g$ on $[a,b]$. Conclude that the subset $C[a,b]$ of continuous functions in $\mathcal{R}^1[a,b]$ is a metric space under the metric $d_1$. 

Reading: Tue Nov.9: Riemann integral, Rudin 6.1-12 with $\alpha(x) = x$
Thu Nov.11: holiday
5. Consider the metric space \((C[-1, 1], d_1)\) from 4(d) (here \(a = -1, b = 1\)).

(a) Let \(f_n\) denote the function

\[
   f_n(x) = \begin{cases} 
   1, & x > \frac{1}{n}, \\
   nx, & 0 \leq x \leq \frac{1}{n}, \\
   0, & x < 0 
   \end{cases}
\]

Show that \(f_n \in C[-1, 1]\), and calculate \(\int_0^1 f_n\).

(b) Show that the sequence \((f_n)\) is a Cauchy sequence in terms of the metric \(d_1\).

(c) Does \(f_n\) converge in \(C[-1, 1]\)? Is this metric space complete?