Reading: Tue Oct.19 : continuity, Rudin 4.1-12

1. (a) Problem 1, page 98 in Rudin
   (b) Problem 3, page 98 in Rudin

2. Let \((X, d)\) be a metric space. Fix \(x_0 \in X\) and a continuous function \(g : \mathbb{R} \to \mathbb{R}\). Show that the function \(X \to \mathbb{R}\) defined by \(x \mapsto g(d(x, x_0))\) is continuous.

3. Let \((S, d_S)\) be a set equipped with the discrete metric (i.e. \(d_S(t, r) = 1\) for \(t \neq r\)).
   (a) Show that any map \(f : S \to X\) into another metric space \(X\) is continuous; using the definition of continuity by sequences.
   (b) Show that any map \(f : S \to X\) into another metric space \(X\) is continuous; using the definition of continuity by \(\varepsilon\)- and \(\delta\)-balls.
   (c) Which maps \(f : \mathbb{R} \to S\) are continuous? (Give an easy characterization and prove it.)

4. Consider the function \(h : \mathbb{Q} \to \mathbb{R}\) given by
   \[
   h(x) = \begin{cases} 
   0 & ; x^2 < 2, \\
   1 & ; x^2 > 2.
   \end{cases}
   \]
   (a) Is \(h\) continuous?
   (b) Can \(h\) be continuously extended to \(\tilde{h} : \mathbb{R} \to \mathbb{R}\)? (I.e. such that \(\tilde{h}(x) = h(x)\) for all \(x \in \mathbb{Q}\).)

5. Prove that the function \(f : \mathbb{C} \to \mathbb{C}, z \mapsto e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n\) is continuous by following the steps below.
   (a) Fix \(T > 0\) and \(\varepsilon > 0\). Show that there exists an \(N \in \mathbb{N}\) such that
   \[
   \sum_{n=N}^{\infty} \frac{t^n}{n!} < \varepsilon \quad \forall t \in [0, T].
   \]
   [Hint: The series for \(e^t\) converges.]
   (b) Show continuity at \(z \in \mathbb{C}\) by splitting
   \[
   e^z - e^x = \sum_{n=1}^{N-1} \frac{1}{n!} (z^n - x^n) + \sum_{n=N}^{\infty} \frac{1}{n!} z^n - \sum_{n=N}^{\infty} \frac{1}{n!} x^n.
   \]
   [Hint: First use (a) with \(T = |z| + 1\), then use the fact that polynomials are continuous.]