The Devil’s Staircase

Recall the usual construction of the Cantor set: $C_0 = [0, 1]$, $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, and in general $C_n$ is a disjoint union of $2^n$ closed intervals, each of length $3^{-n}$, constructed from $C_{n-1}$ by deleting the open-middle-third of each of the $2^{n-1}$ intervals constituting $C_{n-1}$. Then the total length of the intervals in $C_n$ is $(\frac{2}{3})^n$. The Cantor set $C$ is the intersection $C = \bigcap_{n \geq 0} C_n$.

Set $g_n = (\frac{3}{2})^n \mathbb{I}_{C_n}$; that is,

$$g_n(x) = \begin{cases} (\frac{3}{2})^n, & x \in C_n \\ 0, & x \notin C_n. \end{cases}$$

The function $g_n$ is discontinuous only at the $2^{n+1}$ points at the boundaries of the intervals making up $C_n$. This is a finite set, and so $g_n$ is Riemann integrable. So we may define $f_n: [0, 1] \to \mathbb{R}$ as follows:

$$f_n(x) = \int_0^x g_n(t) \, dt.$$  

From the Fundamental Theorem of Calculus, we know that the functions $f_n$ are Lipschitz continuous. Note also that $f_n(0) = 0$, while $f_n(1) = \int_0^1 g_n(t) \, dt$. This integral can be calculated as

$$\int_0^1 g_n(t) \, dt = (\frac{3}{2})^n \int_0^1 \mathbb{I}_{C_n}(t) \, dt = (\frac{3}{2})^n \cdot \text{length}(C_n) = 1.$$  

So $f_n$ is a continuous function with $f_n(0) = 0$ and $f_n(1) = 1$ for each $n$. Notice that $f_n$ is constant on $[0, 1] - C_n$, and is linear with slope $(\frac{3}{2})^n$ on the intervals making up $C_n$. So $f_n$ is monotone increasing. Here is the graph of $f_2$.

![Figure 1. The graph of $f_2$.](image)
Let \( I \) be any one of the \( 2^n \) closed intervals that make up \( C_n \). Then \( g_n(x) = (\frac{3}{2})^n \) for all \( x \in I \), while \( g_{n+1}(x) \) is equal to \( (\frac{3}{2})^{n+1} = \frac{3}{2}g_n(x) \) for \( x \) in the first third or last third of \( I \), and equal to 0 in the middle interval. It follows that

\[
\int_I g_n(t) \, dt = \int_I g_{n+1}(t) \, dt = 2^{-n}.
\]

(*)

Note also that \( g_n(x) = g_{n+1}(x) \) for \( x \notin C_n \). Hence, if \( a \) is any endpoint of an interval in \( C_n \),

\[
\int_a^a g_n(t) \, dt = \int_a^a g_{n+1}(t) \, dt.
\]

(**)

It follows by integrating (*) that

\[
f_{n+1}(x) = f_n(x), \quad x \notin C_n.
\]

(***)

Now, the Cantor set \( C \) is closed, and so any point \( x \notin C \) is contained in a small open interval in \( C^c = \bigcup_{n \geq 0} C_n^c \). Therefore there is some \( N \in \mathbb{N} \) such that \( x \in C_n^c \) for all \( n \geq N \), and by the above we have \( f_n(x) = f_N(x) \) for all \( n \geq N \). This shows that the sequence of functions \( f_n(x) \) converges pointwise on \( C^c \). But it’s even better than that.

Let \( x \in C_n \), and let \( I = [a, b] \) be the interval in \( C_n \) containing \( x \). Applying (**), we have

\[
|f_n(x) - f_{n+1}(x)| = \left| \int_a^x [g_n(t) - g_{n+1}(t)] \, dt \right| = \left| \int_a^x [g_n(t) - g_{n+1}(t)] \, dt \right|
\]

\[
\leq \int_a^x |g_n(t) - g_{n+1}(t)| \, dt
\]

\[
\leq \int_a^b |g_n(t) - g_{n+1}(t)| \, dt.
\]

Again, we have \( |g_n(t) - g_{n+1}(t)| = (\frac{3}{2})^{n+1} - (\frac{3}{2})^n \) on the first and last third of \([a, b]\), while it equals \( (\frac{3}{2})^n \) on the middle third. The difference is therefore bounded above by \( (\frac{3}{2})^{n+1} \) on the interval which has length \( 3^{-n} \), and so the above integral is bounded by \( 3^{-n}(\frac{3}{2})^{n+1} \); that is, we have proved that

\[
|f_n(x) - f_{n+1}(x)| \leq \frac{3}{2} \cdot 2^{-n} < 2^{-n+1}, \quad x \in C_n.
\]

Combining this with (***) we have \( |f_n(x) - f_{n+1}(x)| < 2^{-n+1} \) for all \( x \). It follows that the sequence \( \{f_n\} \) is uniformly Cauchy, and therefore converges uniformly to a limit function \( f \). The functions \( f_n \) are continuous, and so the uniform limit function \( f \) is continuous. Also, for \( x \notin C_n \) the sequence \( \{f_k(x)\}_{k=n}^\infty \) is constant, and therefore \( f_n(x) = f(x) \). But \( f_n \) is constant on \( C_n^c \). Finally, since \( f_n(0) = 0 \) and \( f_n(1) = 1 \) for all \( n \), we have \( f(0) = 0 \) and \( f(1) = 1 \). We have therefore proved the following:

\[
f_n \rightarrow f \text{ uniformly, where } f \text{ is continuous, } f'(x) = 0 \text{ for } x \in C^c, \text{ and } f(0) = 0 \text{ and } f(1) = 1.
\]

A little additional thought shows that the limit function \( f \) is monotone increasing. It is called the Cantor Function or the Devil’s staircase. Its graph is shown in Figure 2. It can actually be described in simple terms. Here is an algorithm for calculating \( f(x) \) for \( x \in [0, 1] \).

- Express \( x \) in base 3, \([x]_3\). (Choose the representation that does not end in 1111 . . .)
• If \([x]_3\) contains any 1s, with the first 1 being at position \(n\): 
\[ [x]_3 = 0.x_1x_2\ldots x_{n-1}1x_{n+1}\ldots, \]
replace the number with \(T(x)\) in ternary 
\[ [T(x)]_3 = 0.x_1x_2\ldots x_{n-1}2. \]
Otherwise, if 
\[ [x]_3 \text{ contains no 1s (i.e. } x \in C), \text{ then } T(x) = x. \]

• Since \(T(x)\) is an even ternary number (i.e. in \(C\)), we can relabel all 2s as 1s. The 0-1 string that remains, re-interpreted in base 2, is \(f(x)\).

For example, \(1/4\) in base 2 is 0.0202020\ldots (isn’t that surprising? \(1/4 \in C!\)). Hence, 
\[ f[(1/4)]_2 = 0.01010101\ldots = 1/3. \]
On the other hand, \([1/5]_3 = 0.012101210\ldots\), hence 
\[ T(1/5)_3 = 0.02, \text{ and so } [f(1/5)]_2 = 0.01, \text{ which is 1/4 in base } 2; \text{ hence, } f(1/5) = 1/4. \]

In general, a non-constant monotone increasing function that is continuous, differentiable almost everywhere with derivative 0, is called **singular**. These functions really point out the necessity of the continuity (and definition everywhere) of the derivative of \(f\) in order for the Fundamental Theorem of Calculus to make sense. (After all, the derivative of \(f\) is 0 almost everywhere, and so \(f \neq f'\) in this case.)

For other fun examples of singular functions, lookup the Wikipedia entries on **Minkowski’s question mark function**, and **Kolmogorov’s circle map**.

One final note: for those who are physically inclined, the 1998 Nobel Prize in Physics was awarded for the discovery and explanation of what is known as the **fractional quantum Hall effect**. This phenomenon is a physical manifestation of a singular function.