Problems:

1) Let \((f_n)\) be the sequence of functions on \(\mathbb{R}\) defined as follows.
\[
f_0(t) = \sin t \quad \text{and} \quad f_{n+1}(t) = \frac{2}{3}f_n(t) + 1 \quad \text{for } n \in \mathbb{N}.
\]
Show that \(f_n \to 3\) uniformly on \(\mathbb{R}\). What can you say if we choose \(f_0(t) = t^2\)?

*Hint:* Consider first the map \(T(x) = \frac{2}{3}x + 1\) on \(\mathbb{R}\).

2) Suppose \(\varphi : [0, \infty) \to \mathbb{R}\) is continuous and satisfies
\[
0 \leq \varphi(t) \leq \frac{t}{2 + t} \quad (t \geq 0).
\]
Define the sequence \((f_n)\) by setting \(f_0(t) = \varphi(t)\) and \(f_{n+1}(t) = \varphi(f_n(t))\) for \(t \geq 0\) and \(n \in \mathbb{N}\). Prove that the series \(F(t) = \sum_{n=0}^{\infty} f_n(t)\) converges for every \(t \geq 0\) and that \(F\) is continuous on \([0, \infty)\).

3) Does \(f(t) = \sum_{k=1}^{\infty} \sin^2(t/k)\) define a differentiable function on \(\mathbb{R}\)?

4) Suppose \((f_n)\) is a sequence of continuous functions such that \(f_n \to f\) uniformly on a set \(E\). Prove that
\[
\lim_{n \to \infty} f_n(x_n) = f(x)
\]
for every sequence of points \(x_n \in E\) such that \(x_n \to x\), and \(x \in E\). Is the converse of this true?

5) Suppose \((f_n)\) is a sequence of real-valued functions that are Riemann-integrable on all compact subintervals of \([0, \infty)\). Assume further that:
   a) \(f_n \to 0\) uniformly on every compact subset of \([0, \infty)\);
   b) \(0 \leq f_n(t) \leq e^{-t}\) for all \(t \geq 0\) and \(n \in \mathbb{N}\).

Prove that
\[
\lim_{n \to \infty} \int_0^{\infty} f_n(t) \, dt = 0,
\]
where the improper integral \(\int_0^{\infty} f_n(t) \, dt\) is defined as \(\lim_{b \to \infty} \int_0^b f_n(t) \, dt\). Moreover, give an explicit example for a sequence \((f_n)\), so that condition b) does not hold and the conclusion above fails.

*Remark:* In fact, one can relax condition a) to “\(f_n \to 0\) pointwise on \([0, \infty)\).” But then the proof (of this “dominated convergence theorem”) becomes by far more involved when using Riemann’s theory of integration.

6) Suppose \((f_n)\) is an equicontinuous sequence of functions on a compact set \(K\), and \(f_n \to f\) pointwise on \(K\). Prove that \(f_n \to f\) uniformly on \(K\).
7) Show that any uniformly bounded sequence of differentiable functions on a compact interval with uniformly bounded derivatives has a convergent subsequence.
(Hint: To apply the Arzela-Ascoli theorem (Thm 7.25) from the book, show that any family $\mathcal{F}$ of real-valued, differentiable functions $f$ defined on $[a, b]$, satisfying $|f'(x)| \leq M$ for some $M$ and all $x \in [a, b]$ and $f \in \mathcal{F}$, must be equicontinuous.)

Extra problems:

1) In class, we have seen that uniform convergence of a sequence of bounded functions on a set $E$ can be equivalently formulated in terms of the metric $d(f, g) = \sup_{x \in E} |f(x) - g(x)|$. That is, we have $d(f_n, f) \to 0$ if and only if $f_n \to f$ uniformly on $E$.

Having this in mind, we could ask whether an analogous statement holds with respect to pointwise convergence. More specifically, is there a metric $d(f, g)$ such that $d(f_n, f) \to 0$ if and only if $f_n \to f$ pointwise on $E$? Surprisingly, it turns out that the answer is NO when, for example, $E = [0, 1]$. We therefore cordially invite you to prove the following theorem.

**Theorem 1.** There is no metric $d(f, g)$ defined on $C([0, 1])$ such that $d(f_n, f) \to 0$ if and only if $f_n \to f$ pointwise on $[0, 1]$.

Before proving this theorem, you may first show the following weaker statement whose proof requires less effort.

**Theorem 2.** There is no norm $\| \cdot \|$ defined on $C([0, 1])$ such that $\|f_n\| \to 0$ if and only if $f_n \to 0$ pointwise on $[0, 1]$.

*Hint (for proof of Theorem 2).* Consider $f_n \in C([0, 1])$, with $n = 1, 2, 3, \ldots$, such that

i) For every $x \in [0, 1]$, there exists $n_0 = n_0(x)$ such that $f_n(x) = 0$ if $n \geq n_0$.

ii) $f_n \not\equiv 0$ for every $n \geq 1$.

(A possible choice is, for instance, given by $f_n(x) = \sin(n\pi x)$ if $x \in [0, 1/n]$, and $f_n(x) = 0$ if $x \in [1/n, 1]$.) By clever choice of a sequence of real-valued numbers $(c_n)$, prove the claim by considering the sequence $(c_nf_n).

*Hint (for proof of Theorem 1).* Assume there is such a metric $d(f, g)$ on $C([0, 1])$. Then $f_n \to 0$ if and only if, for every $k \in \mathbb{N}$, we have that $f_n \in N_{1/k}(0) = \{y \in C([0, 1]) : d(y, 0) < 1/k\}$, except for finitely many $f_n$. Use this fact and the specific choice $g_n(x) = e^{-n|x-x_0|}$ for suitable $x_0 \in [0, 1]$ to show that $g_n \to 0$ pointwise on $[0, 1]$, which is false! (Since $g(x_0) = 1$ for all $n$.)

2) Assume that $(f_n)$ is a sequence of monotonically increasing functions on $\mathbb{R}$ with $0 \leq f_n(x) \leq 1$ for all $x$ and all $n$.

(a) Prove that there is a function $f$ and a sequence $(n_k)$ such that

$$f(x) = \lim_{k \to \infty} f_{n_k}(x)$$

for every $x \in \mathbb{R}$. (This result is usually called Helly’s selection theorem.)

(b) If, moreover, $f$ is continuous, prove that $f_{n_k} \to f$ uniformly on compact sets.

*Hint:* (i) Some subsequence $(f_{n_k})$ converges at all rational points $r$, say, to $f(r)$. (ii) Define $f(x) = \sup_{r \leq x} f(r)$ for any $x \in \mathbb{R}$. (iii) Show that $f_{n_k}(x) \to f(x)$ at every $x$ at which $f$ is continuous. (This is where monotonicity is strongly used.) (iv) A subsequence of $(f_{n_k})$ converges at every point of discontinuity of $f$, since there are at most countably many such points. This outlines the proof of (a). To prove (b), modify your proof of (iii) appropriately.