Solution/Proof of Problem 1. From the definition, we can find that
\[ f_n(t) = \left(\frac{2}{3}\right)^n f_0(t) + \sum_{k=0}^{n-1} \left(\frac{2}{3}\right)^k. \]

Notice that \( \lim_{n \to \infty} \sum_{k=0}^{n-1} \left(\frac{2}{3}\right)^k = \frac{1}{1-2/3} = 3 \) and since \( |f_0(t)| = |\sin t| \leq 1 \) we have
\[ |f_n - 3| \leq |\left(\frac{2}{3}\right)^n| + |\sum_{k=0}^{n-1} \left(\frac{2}{3}\right)^k - 3| \text{ so we have } \forall \epsilon > 0, \]
\begin{itemize}
    \item \( \exists N_1 \text{ s.t. } \forall n > N_1, \left(\frac{2}{3}\right)^n < \frac{\epsilon}{9} \);
    \item \( \exists N_2 \text{ s.t. } \forall n > N_1, |\sum_{k=0}^{n-1} \left(\frac{2}{3}\right)^k - 3| < \frac{\epsilon}{9} \).
\end{itemize}
So take \( N = \max\{N_1, N_2\} \), and we have \( \forall n > N \),
\[ |f_n - 3| \leq |\left(\frac{2}{3}\right)^n| + |\sum_{k=0}^{n-1} \left(\frac{2}{3}\right)^k - 3| \leq \frac{2\epsilon}{3} < \epsilon. \]
So \( f_n \to 3 \) uniformly on \( \mathbb{R} \).

In general, since \( f_n(x) = T^n(f_0(x)) \), \( T \) is a contraction, and the fixed point of \( T \) is 3, we always have pointwise convergence of \( f_n \) to 3. However, from the argument above we see that this is uniform convergence if and only if the function \( f_0 \) is bounded. Thus for \( f_0(t) = t^2 \), the convergence is uniform on any bounded subset of \( \mathbb{R} \), but not on all of \( \mathbb{R} \).

Solution/Proof of Problem 2. Since \( t \geq 0 \), we have \( 0 \leq \phi(t) \leq \frac{1}{2^n} \leq \frac{1}{9^n} \). So we have
\[ 0 \leq f_n(t) = \phi(f_{n-1}(t)) \leq \frac{1}{2} f_n(t) \leq \cdots \leq \frac{1}{2^n} f_0(t) = \frac{1}{2^n} \phi(t) \leq \frac{1}{2^n} \frac{t}{2 + t} \leq \frac{1}{2^n}. \]

From the convergence of \( \sum \frac{1}{2^n} \), we have \( \sum_{n=0}^{\infty} f_n(t) \to F(t) \) uniformly, since each partial sum \( \sum_{n=0}^{\infty} f_n(t) \) is continuous, this implies that \( F \) is continuous.

Solution/Proof of Problem 3. Since differentiability is a local property, we only need to prove that \( f \) is differentiable on every subset \((-s, s) \subset \mathbb{R})
We have
\[ \frac{d}{dt} \sin^2 \left(\frac{t}{k}\right) = 2 \sin \left(\frac{t}{k}\right) \cos \left(\frac{t}{k}\right) = \frac{1}{k} \sin \left(\frac{2t}{k}\right), \]
so if \( F_n(t) = \sum_{k=1}^{n} \sin^2 \left(\frac{t}{k}\right) \), then
\[ \frac{d}{dt} F_n = \sum_{k=1}^{n} \frac{1}{k} \sin \left(\frac{2t}{k}\right). \]
We can use \(|\sin x| \leq |x|\) to see that \(F'_n(t)\) is uniformly Cauchy; indeed, whenever \(n < m\) we have

\[
\|F'_n - F'_m\| = \sup_{t \in [-\pi, \pi]} \left| \sum_{k=m}^{n} \frac{1}{k} \sin\left(\frac{2t}{k}\right) \right| \leq \sup_{t \in [-\pi, \pi]} \left| \sum_{k=m}^{n} \frac{1}{k} \right| \leq \frac{2s}{k^2}
\]

and since \(\sum \frac{1}{k^2}\) converges, we can make this last sum as small as we like. It follows that \(F'_n(t)\) converges uniformly, it’s also clear that \(F_n(0) \to 0\). From Theorem 7.17, we know that \(F_n(t)\) converges to a function \(F(t)\) such that \(F'(t)\) exists and \(F'(t) = \lim_{n \to \infty} F'_n(t)\). So we get the conclusion.

**Solution/Proof of Problem 4.** Since \(f_n \to f\) uniformly, and \(f_n\) are continuous, so is \(f\). Now for any \(\epsilon > 0\), we have

- \(\exists N_1, \text{ s.t. } \forall n > N_1, \text{ and } \forall x \in E, |f(x) - f_n(x)| < \frac{\epsilon}{3}\); 
- \(\exists \delta > 0, \text{ s.t. } \forall|y - x| < \delta, |f(y) - f(x)| < \frac{\epsilon}{3}\); 
- \(\exists N_2, \text{ s.t. } \forall n > N_2, \forall x, y \in E, |x - y| < \delta\).

So we have for \(n > \max\{N_1, N_2\}\),

\[
|f_n(x_n) - f(x)| \leq |f_n(x_n) - f_n(x)| + |f_n(x) - f(x)| \leq \frac{2\epsilon}{3} \leq \epsilon.
\]

So we get the conclusion.

The converse can be formulated different ways. Here’s one that’s true: If \((f_n)\) is a sequence of continuous functions that converge pointwise to a function \(f\) on a compact set \(E\), and \(\lim_{y \to x} f(y)\) always exists, then

\(f_n \to f\) uniformly \(\iff\) \(\lim_{n \to \infty} f_n(x_n) = \lim_{n \to \infty} f(x_n)\) whenever \((x_n)\) converges.

The proof of \(\to\) is above, to prove \(\leftarrow\) assume that \(f_n\) does not converge to \(f\) uniformly. This implies that

for some \(\varepsilon_0 > 0\) and for every \(N \in \mathbb{N}\)

there exists \(n > N\) such that \(\|f_n(x_n) - f(x_n)\| > \varepsilon_0\)

\(\iff\) for some \(\varepsilon_0 > 0\) and for every \(N \in \mathbb{N}\)

there exists \(n > N\) and \(y_n \in E\) such that \(|f_n(y_n) - f(y_n)| > \varepsilon_0\)

Since \(E\) is compact, the sequence \((y_n)\) has a convergent subsequence, which we denote \((x_n)\). Say that \(\lim_{n \to \infty} f(x_n) = L\) and find \(N' \in \mathbb{N}\) such that \(n > N'\) implies \(|f(x_n) - L| < \varepsilon_0/2\). Then, for any \(n > N'\) we have \(|f_n(x_n) - L| > \varepsilon_0/2\) and hence

\[
\lim_{n \to \infty} f_n(x_n) \neq \lim_{n \to \infty} f(x_n),
\]

which proves the converse.

Notice that if we do not require the original sequence to be continuous, then the converse is not true. Take \(E = \mathbb{R}\). Consider a sequence of functions

\[
f_n(x) = \begin{cases} 
0 & x \in (-n, n) \\
1 & \text{otherwise}
\end{cases}
\]

Then \(f_n\) converge to 0 pointwise and \(f_n\) does not converge uniformly to 0. But we can easily see that for any convergent sequence \(\{x_n\}\), \(f_n(x_n) \to 0\).
Solution/Proof of Problem 5. Form condition (b), we have
\[ 0 \leq \int_0^\infty f_n(t)\,dt \leq \int_0^\infty e^{-t} = 1. \]
So we have
\[ 0 \leq \lim_{T \to \infty} \int_T^\infty f_n(t)\,dt \leq \lim_{T \to \infty} \int_T^\infty e^{-t} = 0. \]
So for \( \varepsilon > 0, \exists S \) s.t. \( \forall n \)
\[ 0 \leq \int_S^\infty f_n(t)\,dt \leq \int_S^\infty e^{-t} \leq \frac{\varepsilon}{3}. \]
On the other hand, from condition (a), we have for \( \frac{\varepsilon}{3S} > 0, \exists N \) s.t. \( \forall n > N \)
\[ \int_0^S f_n(t)\,dt \leq \int_0^S \frac{\varepsilon}{3S} \,dt = \frac{\varepsilon}{3}. \]
So we have for \( \varepsilon > 0, \exists N \) s.t. \( \forall n > N \)
\[ \int_0^\infty f_n(t)\,dt \leq \int_0^S f_n(t)\,dt + \int_S^\infty f_n(t)\,dt \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon. \]
So \( \lim_{n \to \infty} \int_0^\infty f_n(t)\,dt = 0. \)
Condition (b) is necessary. In fact, we can consider \( f_n(t) = \frac{1}{nt} \), which satisfies condition (a). It is clear that the conclusion does not hold.

Solution/Proof of Problem 6. Since \( \{f_n\} \) is equicontinuous, so for any \( \frac{\varepsilon}{3} > 0, \exists \delta > 0 \) s.t. for any two points \( x, y \in K \), if \( |x - y| < \delta \), then \( |f_n(x) - f_n(y)| < \frac{\varepsilon}{3} \).

Now consider an open covering \( K = \bigcup_{x \in K} D_\delta(x) \) where \( D_\delta(x) \) is a disc with center \( x \) and radius \( \delta \). Since \( K \) is compact, we can find finite disc to cover \( K \). Let \( K = \bigcup_{i=1}^\infty D_\delta(x_i) \).

For any \( x \in K \), we have \( x \in D_\delta(x_i) \) for some \( x_i \). So \( |f_n(x) - f_n(x_i)| < \frac{\varepsilon}{3} \).

For each \( i \), \( f_n(x_i) \to f(x_i) \), we have for \( \frac{\varepsilon}{3} > 0, \exists N_i > 0 \), s.t. \( \forall n > N_i, |f_n(x_i) - f(x_i)| < \frac{\varepsilon}{3} \).

Let \( N = \max_i N_i \), so \( \forall n > N, |f_n(x_i) - f(x_i)| < \frac{\varepsilon}{3} \) for any \( i \).

So \( \forall n, n > N \)
\[ |f_n(x) - f_n(x_i)| = |f_n(x) - f_n(x_i) + f_n(x_i) - f(x_i)| + |f(x_i) - f_n(x)| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \leq \varepsilon. \]
So \( f_n \to f \) uniformly.

Solution/Proof of Problem 7. Since \( f_n \) is uniformly bounded, so there exists \( M \) s.t. \( |f_n'(x)| \leq M, \forall x, n \).

For any \( x \leq y \), by MVT, we have \( \exists \xi \in [x, y] \) s.t.
\[ |f_n(x) - f_n(y)| = |f_n'\!(\xi)(x - y)| \leq M|x - y|. \]
So for any \( \varepsilon > 0, \exists \delta = \varepsilon/M > 0 \) s.t. \( \forall x, y, |x - y| < \delta, |f_n(x) - f_n(y)| < \varepsilon. \)
So \( f_n \) is equicontinuous. Then from Arzela-Ascoli theorem, we got the conclusion.
18.100B Analysis I
Fall 2010

For information about citing these materials or our Terms of Use, visit: http://ocw.mit.edu/terms.